Electromagnetics

Lecture Notes

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Review

Vector Analysis

And

Coordinate Systems

Contents

Chapter 1

INTRODUCTION

1.1 INTRODUCTION AND MOTIVATION:

The focus is on electricity and magnetism, including electric fields, magnetic fields, electromagnetic forces, conductors and dielectrics.

Electromagnetics is the study of electric and magnetic phenomena caused by electrical charges at rest or in motion.It is one of the most important courses in electrical engineering. It can also be regarded as the study of the interaction between electrical charges at rest and in motion. It is a branch of electrical engineering or physics in which electrical and magnetic phenomena are studied.

Mobile phone communication can not be explained by circuit theory concepts alone. The source feeds into an open circuit because the upper tip of the antenna is not connected to any thing physically, hence no current will flow and nothing will happen.This cannot explain why communication can be established between moving telephone units.

Since the beginning of the twentieth century,the study of electricity and magnetism has been in its mature stage of development. A steady but ever slower accretion of knowledge has taken place, so that the graph is asymptotically approaching a plateau.

When a body of knowledge is in this stage it is called classical. It should not be inferred from this that a classical subject is one that is at best fruitless. There are still many unsolved problems in electricity and magnetism

1. Why there are two kinds of charge . Only one kind of gravitational mass has been found so far. Is mass a simpler and more fundamental property?

2. Must charge always be associated with mass? Mass always is not associated with charge.

3. Charge is quantized. Why does the minimum quantity of charge have the value it does have?

4. Why are electrical forces so overwhelmingly larger than gravitational forces?

Electrical force is 10^{43} times the gravitational force.

The commonly held view by students, expressed vehemently , particularly recent survivors of the course is , that electromagnetics is difficult, complicated, and a mysterious discipline. It requires mastery of abstruse mathematical techniques,. It also entails juggling a bewildering variety of equations , laws and rules, they decide. Even an intense study has left them with only superficial grasp of the concepts.

Few see the beauty of electromagnetics: not many appreciate the simplicity and and economy of its fundamental laws. A minority realize its wide ranging utility, the breadth and scope of its applications. Only a minority master it enough to be able to use its principles to understand or predict the capabilities and limitations of the engineering systems they need to analyze or design.

1.2 A NOTE TO THE STUDENT:

1. Pay particular attention to vector analysis, the mathematical tool for this course.

2. Do not attempt to memorize too many formulae . Try to understand how the formulae are related.

3. Try to identify the key words or terms in a given definition or law

4. **Attempt to solve as many problems as possible. Practice is the best way to gain skill**

Your brain should think that what you want to learn is important. It is built to search, scan and wait for something to happen. It is built that way and helps you to stay alive. You should know what is important and what is not important. So when you want to learn subject you should know that you have to study and concentrate whether you like it or not.
What does it take to learn something? First you have to get it, and make sure that yo do not forget it. pushing facts mechanically into the head does not help. Learning is a lot more than text on a page. The following are the principles of good learning:

1. Get-and keep - the reader's attention
2. Use a conversational and personalized style
3. Touch their emotions
4. Make it visual
5. Get the learner to think more deeply

Thinking about thinking:

Real learning takes place , that too quickly and more deeply, if you ***pay attention to how you pay attention.Think about how you think. Learn how you learn.***

Nobody takes a course on learning. You are expected to learn, but rarely taught to learn. The trick is to get the brain think that what you are going to learn is important.

Ten Principles to bend the Brain:

1. ***Slow down****. If you slow down you understand well. The more you understand , the less you have to memorize.*

2. ***Do the exercises.*** *Write your own notes. Use pencil. Physical activity while learning can increase learning.*

3. ***Do not Do all Your Learning in One Place****. Stand up,stretch, move around , change chairs, change rooms.*

4. ***Make what you want to learn the last thing that you read before bed, or at least the last challenging thing.*** *Part of the learning (especially the long term memory) happens after you put down your book down. Brain needs time for processing, so if you put something new, you loose some of what you just learned.*

5. ***Drink lots of water.*** *Dehydration decreases cognitive function.*

6. ***Talk about it and also loudly.*** *Better try to explain it to someone else loudly.*

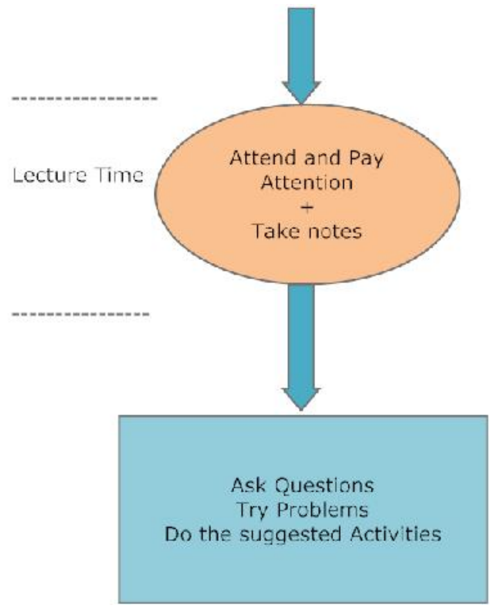
7. ***Listen to your brain .*** *Know when the brain is overloaded.*

8. ***Feel something.*** *Get involved. Feeling nothing at all is bad.*

9. ***There are no dumb questions.*** *Sometimes the questions are more useful than the answers.*

10. ***Shut your mouth and listen, suspending your judgment****, when you want to learn something from some other person.*

- Review the previous lecture
- Ask questions if needed
- Do suggested course activities



1.3 APPLICATIONS OF ELECTROMAGNETIC FIELD THEORY:

1. Microwaves
2. Antennas
3. Electrical Machines
4. Satellite Communications
5. Plasmas
6. Fiber Optics
7. Bio-electromagnetics
8. Nuclear Research
9. Electro mechanical energy conversion
10. Radar
11. Meteorology
12. Remote sensing
13. Induction Heating
14. Surface Hardening, Dielectric Heating
15. Enhance Vegetable Taste by Reducing Acidity
16. Speed baking Of Bread
17. Physics based signal processing
18. Computer chip design
19. lasers
20. EMC/EMI Analysis

Chapter 2

REVIEW

2.1 REVIEW OF COORDINATE SYSTEMS AND VECTOR CALCULUS:

2.1.1 Learning Objectives

- To be able to describe the three coordinate systems we use in describing fields: Cartesian, cylindrical, and spherical.
- To be able to manipulate vectors and perform common operations with them: decomposition, addition, subtraction, dot products, and cross products.
- To be able to describe the fundamental meaning of integration in one, two, and three dimensions as a summation process.
- To be able to describe the fundamental meaning of differentiation.
- To be able to recognize situations where Taylor series should be used, and to be able to demonstrate that you can carry out a Taylor series expansion to first order.
- To be able to understand the significance of the gradient, divergence , and curl operations and prove divergence and Stoke's therorems.

2.1.2 Introduction:

In order to be able to handle with ease many of the elctromagnetic quantities which are vectors, we must choose a coordinate system. we will see how to resolve a given vector into components in these coordinate systems and how to transform a vector from one coordinate system into another.

We will discuss the significance of the gradient, divergence , and curl operations and prove divergence and Stoke's theorems.

This chapter discusses about vector analysis which consists of

1. Vector algebra - addition, subtraction , and multiplication of vectors
2. Vector calculus - differentiation and integration of vectors; gradient,divergence, and curl operations.

This chapter discusses also about

1. Orthogonal coordinate systems- Cartesian, Cylindrical, and spherical coordinates

2.1.3 COORDINATE SYSTEMS:

The dimension of space comes from nature. The measurement of space comes from us. The laws of electromagnetics are independent of a particular coordinate system. However application of the laws to the solution of a particular problem imposes the need to use a suitable coordinate system. It is the shape of the boundary that determines the most suitable coordinate system to use in its solution. To represent points in space we need a coordinate system. The co-ordinate system may be orthogonal or non-orthogonal. Coordinate systems can also be right handed or left handed. We discuss only right handed, orthogonal coordinate systems. The coordinate systems discussed are three dimensional coordinate systems. The coordinate systems are defined by a set of planes and/or surfaces. A coordinate system defines a set of three reference directions at each and every point in space.The origin of the coordinate system is the reference point relative to which we locate every other point in space. A position vector defines the position of a point in space relative to the origin.The three reference directions are referred to as coordinate directions. Unit vectors along the coordinate directions are called base vectors. In any three dimensional coordinate system , an arbitrary vector can be expressed in terms of a superposition of the three base vectors. Different coordinate systems are different ways to measure space.

2.1.3.1 RECTANGULAR CARTESIAN COORDINATE SYSTEM:

The rectangular Cartesian coordinate system is described by three planes which are mutually perpendicular to each other. The three planes intersect at a point 'O' which is called the origin of the coordinate system. There are three coordinate axes which are usually denoted by x,y,z. Values of x,y,z are measured from the origin. The three planes are

- x= constant plane ie yz plane
- y= constant plane ie xz plane
- z= constant plane ie xy plane

A point in rectangular coordinate system is defined by (x, y, z). The limits for the coordinates are

$$-\infty \leq x \leq \infty \tag{2.1}$$
$$-\infty \leq y \leq \infty \tag{2.2}$$
$$-\infty \leq z \leq \infty \tag{2.3}$$

The unit or base vectors are a_x, a_y, a_z. The following relations hold for the dot and cross products of the unit vectors

$$a_x \times a_y = a_z \tag{2.4}$$
$$a_y \times a_z = a_x \tag{2.5}$$
$$a_z \times a_x = a_y \tag{2.6}$$

$$a_x \bullet a_y = 0 \tag{2.7}$$
$$a_y \bullet a_z = 0 \tag{2.8}$$
$$a_z \bullet a_x = 0 \tag{2.9}$$

The differential length element is given by

$$dl = dxa_x + dya_y + dza_z \tag{2.10}$$

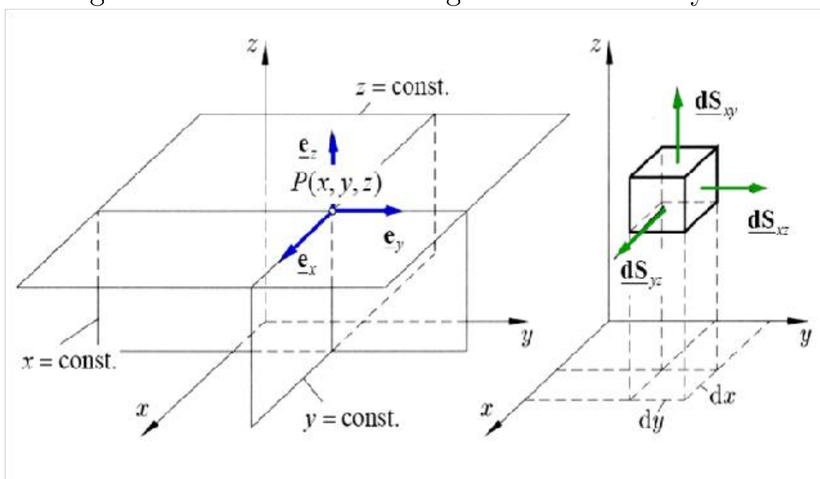
The differential area elements are

$$ds_{rec} = \left\{ \begin{matrix} dxdya_z \\ dydza_x \\ dzdxa_y \end{matrix} \right\}$$

the differential volume element is

$$dv = dxdydz \tag{2.11}$$

Figure 2.1: Cartesian rectangular coordinate system



2.1.3.2 CYLINDRICAL COORDINATE SYSTEM:

The cylindrical coordinate system is also defined by three mutually orthogonal surfaces. They are a cylinder and two planes. One of the planes is the same as the $z = constant$ plane in the Cartesian coordinate system. The second plane is orthogonal to the $z = constant$ plane and hence contains the $z-axis$. it makes an angle ϕ with the $xz-plane$. This plane is defined by $\phi = constant$.The third one is a cylinder whose axis is the z axis and has a radius $\rho = constant$ from the $z - axis$. So a point in cylindrical coordinates is defined by (ρ, ϕ, z). The limits for the coordinates are

$$\begin{aligned} 0 &\leq \rho \leq \infty \\ 0 &\leq \phi \leq 2\pi \\ -\infty &\leq z \leq \infty \end{aligned} \tag{2.12}$$

The unit or base vectors are $a_\rho, a_\phi \, and \, a_z$. The following relations hold for the dot and cross products

$$\begin{aligned} a_\rho &\times a_\phi = a_z \\ a_\phi &\times a_z = a_\rho \\ a_z &\times a_\rho = a_\phi \end{aligned} \tag{2.13}$$

$$\begin{aligned} a_\rho &\bullet a_\phi = 0 \\ a_\phi &\bullet a_z = 0 \\ a_z &\bullet a_\rho = 0 \end{aligned} \tag{2.14}$$

The vector differential length element is given by

$$dl_{cy} = d\rho a_\rho + \rho d\phi a_\phi + dz a_z \tag{2.15}$$

The three differential area elements are

$$ds_{cy} = \begin{Bmatrix} (\rho d\phi) & (dz) & a_\rho \\ (dz) & (d\rho) & a_\phi \\ (d\rho) & (\rho d\phi) & a_z \end{Bmatrix} \tag{2.16}$$

See the fig. The area is $\rho d\phi dz a_\rho$

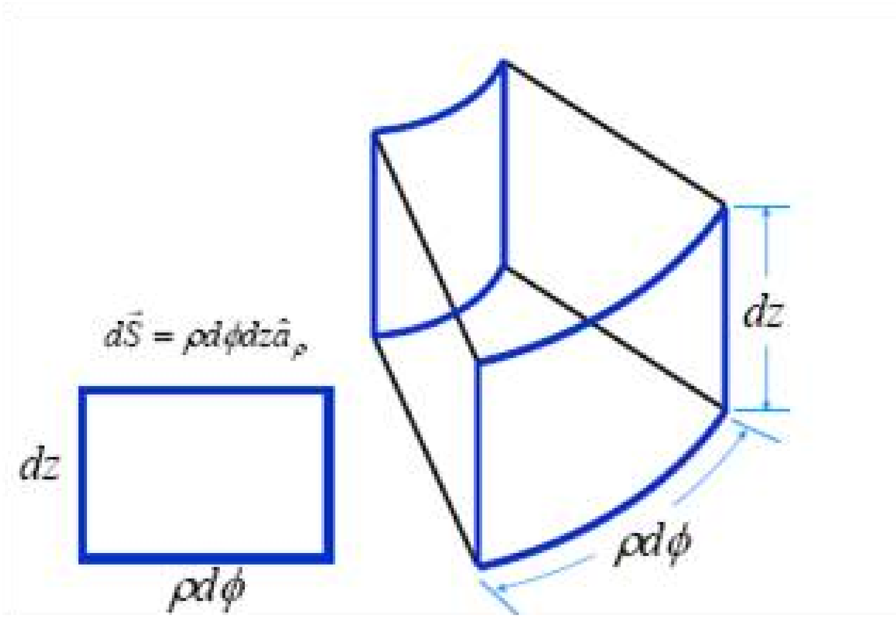


Figure 2.2: Cylindrical area

See the fig. below. The area is $d\rho dz a_\phi$

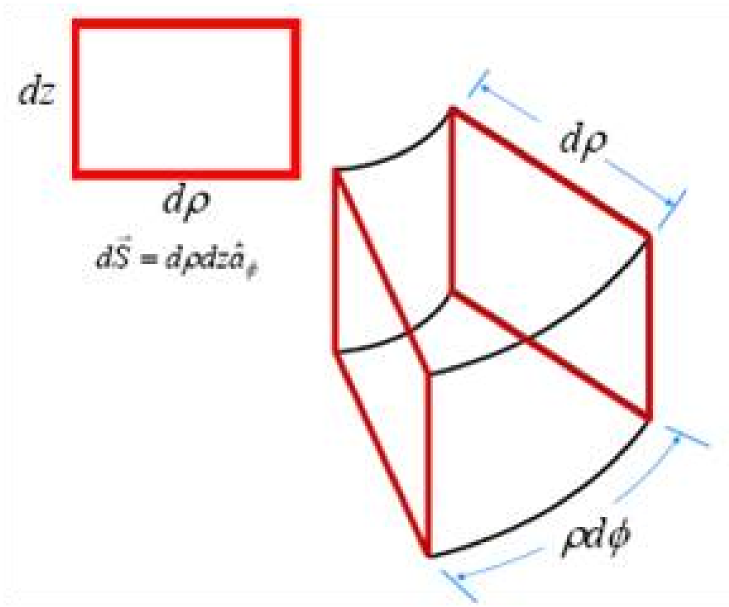


Figure 2.3: Cylindrical area element

See the fig. below. The area element is $\rho d\rho d\phi a_z$

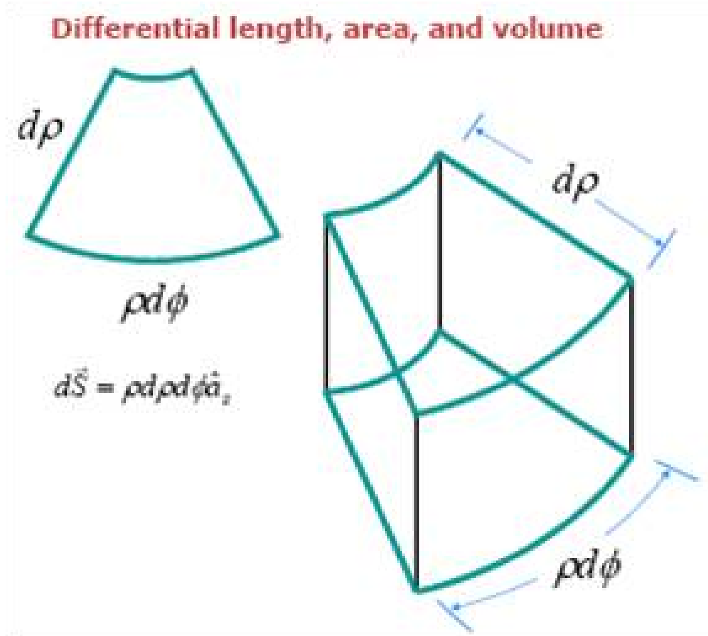


Figure 2.4: Cylindrical area element

The differential volume element is given by

$$dv_{cy} = \rho d\rho d\phi dz \tag{2.17}$$

See fig

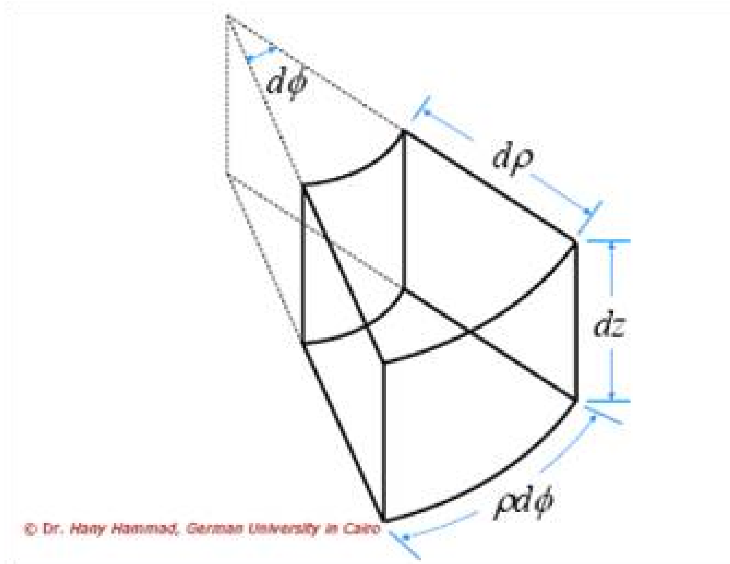


Figure 2.5: Volume element in cylindrical coordinate system

Another view of cylindrical coordinate system:

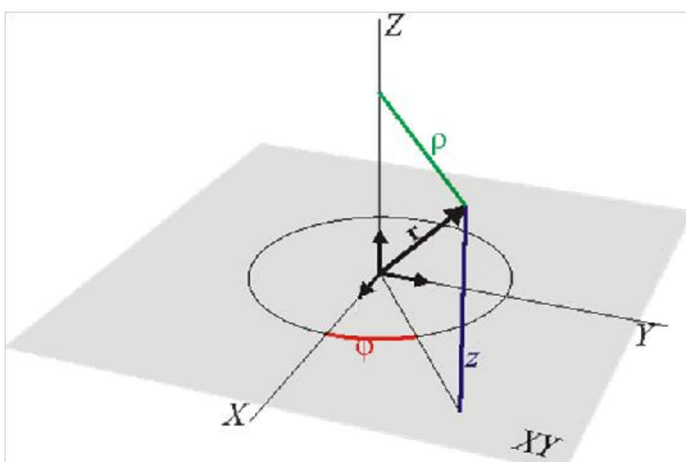


Figure 2.6: Cylindrical coordinate system

A third view of cylindrical coordinate system:

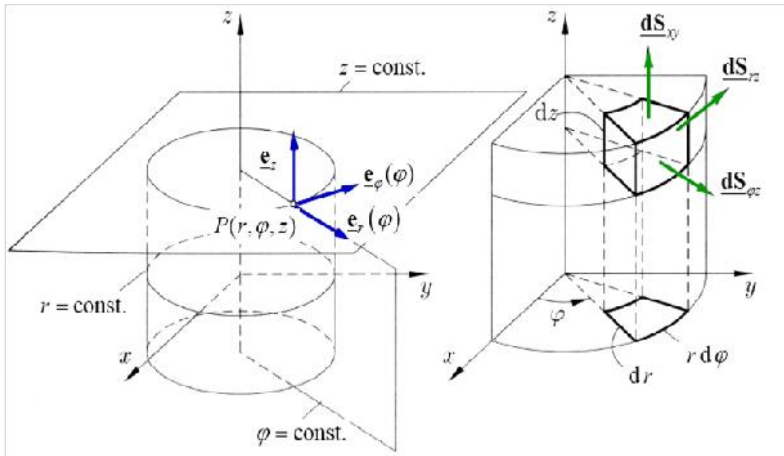


Figure 2.7: Cylindrical coordinate system

2.1.3.3 SPHERICAL COORDINATE SYSTEM:

It is defined by two surfaces and one plane. The surfaces are a sphere and a cone. The plane is $\phi = constant$ plane. The sphere is centered at the origin and has a radius $r = constant$. The cone has its vertex at the origin and its surface is symmetrical about the $z-axis$, so that the angle θ which the conical surface makes with the $z-axis$ is constant. A point in spherical co-ordinates is represented by $p = (r, \theta\, and\ \phi)$. The limits for the coordinates are

$$\begin{aligned} 0 &\leq r \leq \infty \\ 0 &\leq \theta \leq \pi \\ 0 &\leq \phi \leq 2\pi \end{aligned} \tag{2.18}$$

The differential length is given by

$$dl_{sph} = dra_r + rd\theta a_\theta + rsin\theta d\phi a_\phi \tag{2.19}$$

The differential area element is given by

$$ds_{sph} = \left\{ \begin{array}{c} rdrd\theta a_\phi \\ r^2 \sin\theta d\theta d\phi a_r \\ r\sin\theta drd\phi a_\theta \end{array} \right\} \tag{2.20}$$

The differential volume element is given by

$$dv_{sph} = r^2 \sin\theta drd\theta d\phi \tag{2.21}$$

See the fig. below for the volume and area elements

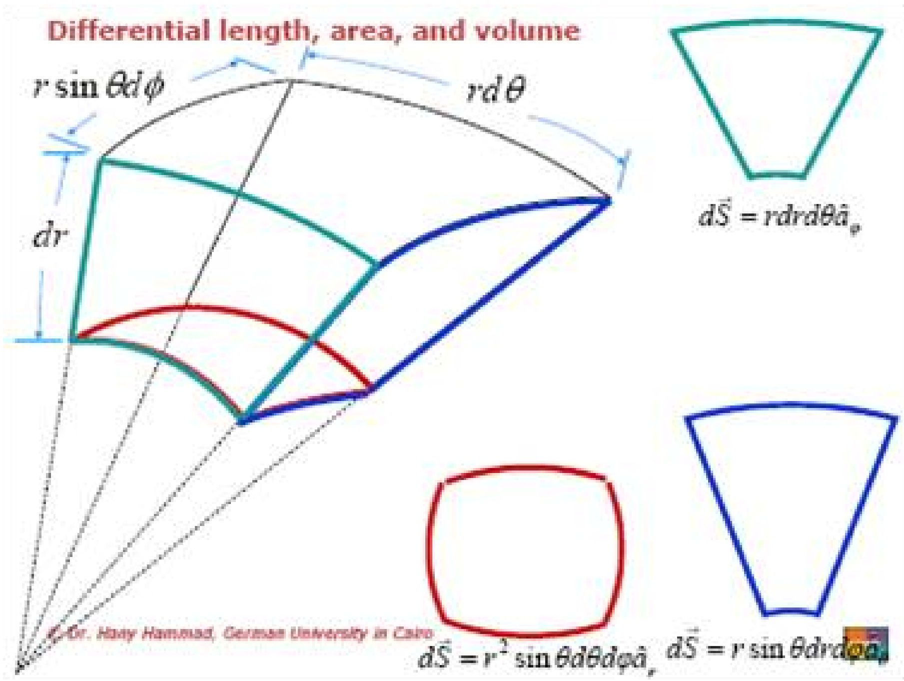
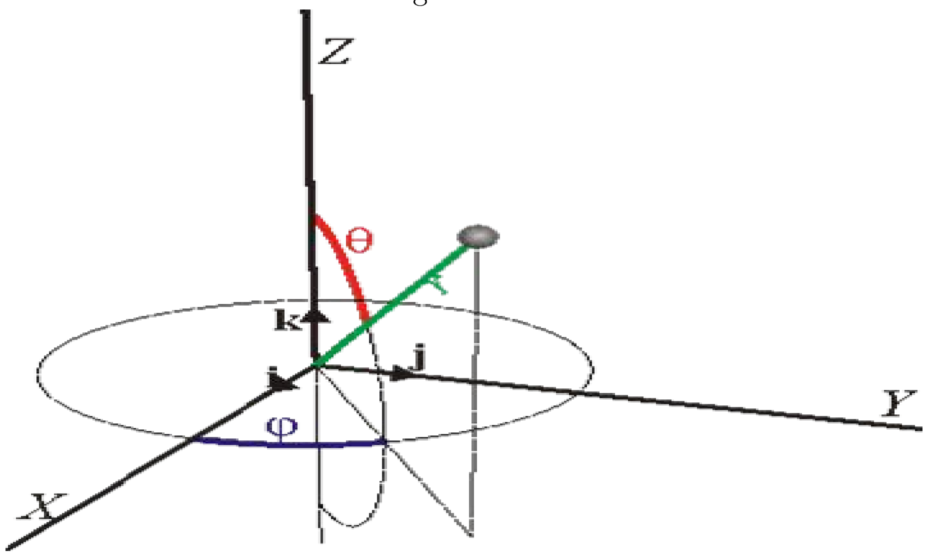


Figure 2.8: Spherical volume and area elements

Spherical coordinate system

Figure 2.9:



Another view of the spherical coordinate system:

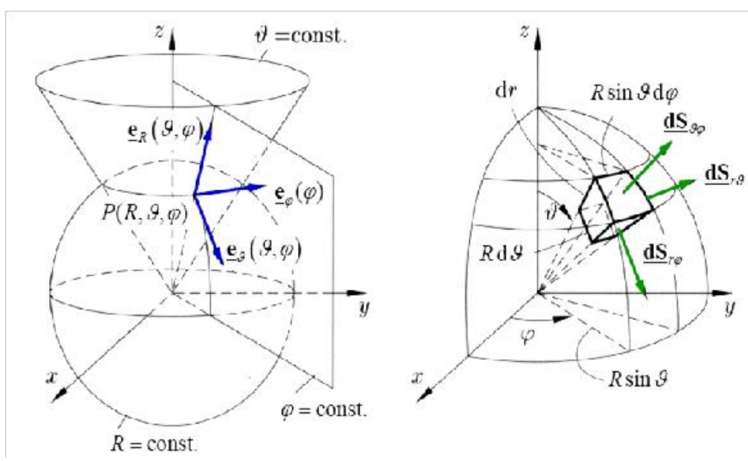


Figure 2.10: Spherical coordinate system

	Cartesian	Cylindrical	Spherical
Orthogonal Surfaces	Three Planes	A Cylinder and two Planes	A Sphere , a Cone , and a Pl
Geometry	Fig.	Fig.	Fig.
Coordinates	x, y, z	ρ, ϕ, z	r, θ, ϕ
Limits Of Coordinates	$-\infty \leq x \leq \infty$ $-\infty \leq y \leq \infty$ $-\infty \leq z \leq \infty$	$0 \leq \rho \leq \infty$ $0 \leq \phi \leq 2\pi$ $-\infty \leq z \leq \infty$	$0 \leq r \leq \infty$ $0 \leq \theta \leq \pi$ $0 \leq \phi \leq 2\pi$
Differential Length elements	$dxa_x + dya_y + dza_z$	$d\rho a_\rho + \rho d\phi a_\phi + dza_z$	$dra_r + rd\theta a_\theta + r\sin\theta d\phi a$
Differential Areas	$dxdya_z$ $dydza_x$ $dzdxa_y$	$\rho d\rho d\phi a_z$ $\rho d\phi dza_\rho$ $d\rho dza_\phi$	$rdrd\theta a_\phi$ $r^2\sin\theta d\theta d\phi a_r$ $r\sin\theta drd\phi a_\theta$
Differential volume	$dxdydz$	$\rho d\rho d\phi dz$	$r^2\sin\theta drd\theta d\phi$

Table 2.1: Summary of Cartesian, Cylindrical and spherical coordinate systems

Dot products of vectors at a point (r, θ, ϕ)

	a_x	a_y	a_z	a_ρ	a_ϕ	a_r	a_θ	a_ϕ
a_x	1	0	0	$\cos\phi$	$-\sin\phi$	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
a_y		1	0	$\sin\phi$	$\cos\phi$	$\sin\theta\sin\phi$	$\cos\theta\cos\phi$	$\cos\phi$
a_z			1	0	0	$\cos\theta$	$-\sin\theta$	0
a_ρ				1	0	$\sin\theta$	$\cos\theta$	0
a_ϕ					1	0	0	1
a_r						1	0	0
a_θ							1	0

Table 2.2: Dot products of unit vectors at a point

Cross product of unit vectors at a point (r, θ, ϕ)

	a_x	a_y	a_z	a_ρ	a_ϕ	a_r	a_θ
a_x	0	a_z	$-a_y$	$\sin\phi a_z$	$\cos\phi a_z$	$\sin\theta\sin\phi a_z - \cos\theta a_y$	$\cos\theta\sin\phi a_z + \sin\theta a_y$
a_y		0	a_x	$-\cos\phi a_z$	$\sin\phi a_z$	$-\sin\theta\cos\phi a_z + \cos\theta a_x$	$-\cos\theta\cos\phi a_z - \sin\theta a_x$
a_z			0	a_ϕ	$-a_\rho$	$\sin\theta a_\phi$	$\cos\theta a_\phi$
a_ρ				0	a_z	$-\cos\theta a_\phi$	$\sin\theta a_\phi$
a_ϕ					0	$-\sin\theta a_z + \cos\theta a_\rho$	$-\cos\theta a_z - \sin\theta a_\rho$
a_r						0	a_ϕ
a_θ							0

Table 2.3: Cross products of unit vectors at a point

2.1.4 TRANSFORMATION OF COORDINATES:

2.1.4.1 CARTESIAN TO CYLINDRICAL:

If a vector is expressed in Cartesian coordinates as $A = A_x a_x + A_y a_y + A_z a_z$

$$x = \rho\cos\phi$$
$$y = \rho\sin\phi \tag{2.22}$$
$$z = z$$

$$\rho = \sqrt{x^2 + y^2}$$
$$\phi = \arctan \tfrac{y}{x} \tag{2.23}$$
$$z = z$$

then the equivalent vector in cylindrical coordinates is given by

$$A_\rho = A_x \cos\phi + A_y \sin\phi$$
$$A_\phi = -A_x \sin\phi + A_y \cos\phi \tag{2.24}$$
$$A_z = A_z$$

2.1.4.2 CARTESIAN TO SPHERICAL:

$$A_r = A_x \sin\theta \cos\phi + A_y \sin\theta \sin\phi + A_z \cos\theta$$
$$A_\theta = A_x \cos\theta \cos\phi + A_y \cos\theta \sin\phi - A_z \sin\theta \tag{2.25}$$
$$A_\phi = -A_x \sin\phi + A_y \cos\phi$$

$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \arccos \tfrac{z}{\sqrt{x^2+y^2+z^2}} \tag{2.26}$$
$$\phi = \arctan \tfrac{y}{x}$$

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi \tag{2.27}$$
$$z = r\cos\theta$$

2.1.4.3 CYLINDRICAL TO CARTESIAN:

$$x = \rho\cos\phi$$
$$y = \rho\sin\phi \tag{2.28}$$
$$z = z$$

$$A_x = \frac{A_\rho x - A_\phi y}{\sqrt{x^2+y^2}}$$
$$A_y = \frac{A_\rho y + A_\phi x}{\sqrt{x^2+y^2}} \tag{2.29}$$
$$A_z = A_z$$

2.1.4.4 CYLINDRICAL TO SPHERICAL:

$$r = \sqrt{\rho^2 + z^2}$$
$$\theta = \arctan\frac{\rho}{z} \tag{2.30}$$
$$\phi = \phi$$

$$A_r = A_\rho \sin\theta + A_z \cos\theta$$
$$A_\theta = A_\rho \cos\theta - A_z \sin\theta \tag{2.31}$$
$$A_\phi = A_\phi$$

where

$$\cos\theta = \frac{z}{\sqrt{\rho^2+z^2}}$$
$$\sin\theta = \frac{\rho}{\sqrt{\rho^2+z^2}} \tag{2.32}$$

2.1.4.5 SPHERICAL TO CARTESIAN:

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\cos\phi \tag{2.33}$$
$$z = r\cos\theta$$

$$A_x = \frac{A_r x\sqrt{x^2+y^2} + A_\theta xz - A_\phi y\sqrt{x^2+y^2+z^2}}{\sqrt{(x^2+y^2)(x^2+y^2+z^2)}}$$
$$A_y = \frac{A_r y\sqrt{x^2+y^2} + A_\theta yz + A_\phi x\sqrt{x^2+y^2+z^2}}{\sqrt{(x^2+y^2)(x^2+y^2+z^2)}} \tag{2.34}$$
$$A_z = \frac{A_r z - A_\theta\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$$

2.1.4.6 SPHERICAL TO CYLINDRICAL:

$$\rho = r\sin\theta$$

$$\phi = \phi \tag{2.35}$$

$$z = r\cos\theta$$

$$A_\rho = \frac{A_r r\sin\theta + A_\theta z}{\sqrt{r^2\sin^2\theta + z^2}}$$

$$A_\phi = A_\phi \tag{2.36}$$

$$A_z = \frac{A_r z - A_\theta r\sin\theta}{\sqrt{r^2\sin^2\theta + z^2}}$$

2.1.4.7 COORDINATE TRANSFORMATIONS IN MATRIX FORM:

Rectangular to cylindrical:

$$\begin{bmatrix} A_\rho \\ A\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \tag{2.37}$$

Rectangular to Spherical:

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & -\cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \tag{2.38}$$

Cylindrical to rectangular:

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & -\frac{y}{\sqrt{x^2+y^2}} & 0 \\ \frac{y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} \tag{2.39}$$

Cylindrical to spherical:

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} \tag{2.40}$$

Spherical to rectangular:

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & -\cos\theta\sin\phi & 0-\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} \tag{2.41}$$

Spherical to cylindrical:

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \frac{\rho}{\sqrt{\rho^2+z^2}} & \frac{z}{\sqrt{\rho^2+z^2}} & 0 \\ 0 & 0 & 1 \\ \frac{z}{\sqrt{\rho^2+z^2}} & \frac{\rho}{\sqrt{\rho^2+z^2}} & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} \tag{2.42}$$

2.2 COORDINATE COMPONENT TRANSFORMATIONS:

Rectangular to Cylindrical

$$
\begin{aligned}
x &= \rho\cos\phi \\
y &= \rho\sin\phi \\
z &= z
\end{aligned}
$$

$$
\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}
$$

Rectangular to Spherical

$$
\begin{aligned}
x &= r\sin\theta\cos\phi \\
y &= r\sin\theta\sin\phi \\
z &= r\cos\theta
\end{aligned}
$$

$$
\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}
$$

Table 2.4: Rectangular to cylindrical and spherical

Cylindrical to Rectangular:

$$\begin{aligned} \rho &= \sqrt{x^2+y^2} \\ \phi &= \arctan\frac{y}{x} \\ z &= z \end{aligned}$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & -\frac{y}{\sqrt{x^2+y^2}} & 0 \\ \frac{y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

Cylindrical to spherical:

$$\begin{aligned} \rho &= r\sin\theta \\ \phi &= \phi \\ z &= r\cos\theta \end{aligned}$$

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

Table 2.5: Cylindrical to Rectangular and Spherical

Spherical to Rectangular:

$$
\begin{aligned}
r &= \sqrt{x^2 + y^2 + z^2} \\
\theta &= \arctan \frac{\sqrt{x^2 + y^2}}{z} \\
\phi &= \arctan \frac{y}{x}
\end{aligned}
$$

$$
\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2+z^2}} & \frac{xz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} & -\frac{y}{\sqrt{x^2+y^2}} \\ \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{yz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} & \frac{x}{\sqrt{x^2+y2}} \\ \frac{z}{\sqrt{x^2+y^2+z^2}} & -\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}
$$

Spherical to Cylindrical:

$$
\begin{aligned}
r &= \sqrt{\rho^2 + z^2} \\
\theta &= \arctan \frac{\rho}{z} \\
\phi &= \phi
\end{aligned}
$$

$$
\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \frac{\rho}{\sqrt{\rho^2+z^2}} & \frac{z}{\sqrt{\rho^2+z^2}} & 0 \\ 0 & 0 & 1 \\ \frac{z}{\sqrt{\rho^2+z^2}} & -\frac{\rho}{\sqrt{\rho^2+z^2}} & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}
$$

Table 2.6: Spherical to Rectangular and Cylindrical

2.2.0.1 COORDINATE TRANSFORMATION PROCEDURE:

1. Transform the component scalars into the new coordinate system
2. Insert the component scalars into the coordinate transformation matrix and evaluate
3. steps 1 and 2 can be performed in either order

2.3 PARTIAL DERIVATIVES OF UNIT VECTORS:

(All derivatives not listed in the table are zero)

	∂x	∂y	∂z	$\partial \rho$	$\partial \phi$
$\partial a_\rho /$	$-\frac{\sin\phi}{\rho} a_\phi$	$\frac{\cos\phi}{\rho} a_\phi$	0	0	a_ϕ
$\partial a_\phi /$	$\frac{\sin\phi}{\rho} a_\rho$	$-\frac{\cos\phi}{\rho} a_r$	0	0	$-a_\rho$
$\partial a_r /$	$\frac{1}{r}(-\sin\phi a_\phi + \cos\phi a_\phi)$	$\frac{1}{r}(\cos\phi a_\phi + \cos\theta \sin\phi a_\theta)$	$\frac{-\sin\theta}{r} a_\theta$	$\frac{\cos\theta}{r} a_\theta$	$\sin\theta a_\phi$
$\partial a_\theta /$	$\frac{\cot\theta}{r}(-\sin\phi a_\phi - \sin\theta \cos\phi a_r)$	$\frac{\cot\theta}{r}(\cos\phi a_\phi - \sin\theta \sin\phi a_r$	$\frac{\sin\theta}{r} a_r$	$\frac{-\cos\theta}{r} a_r$	$\cos\theta a_\phi$

Table 2.7: Partial deviates of unit vectors

Example:

1. Transform each of the following vectors to cylindrical coordinates at the point specified
   (a) $5a_x$ at $P(\rho = 4, \phi = 120^0, z = 2)$

(b) $5a_x$ at $Q(x = 3, y = 4, z = -1)$

(c) $A = 4a_x - 2a_y - 4a_z$ at $Q(2, 3, 5)$

Ans:

a) The ρ component is $5a_x \bullet a_\rho = 5\cos\phi$

The ϕ component is $5a_x \bullet a_\phi = -5\sin\phi$

The z component is $5a_x \bullet a_z = 0$

so $P = 5\cos\phi a_\rho - 5\sin\phi a_\phi$where $\phi = 120^0$

$$\boxed{P_{cyl} = -2.5a_\rho - 4.33a_\phi}$$

b) $Q = 5\cos\phi a_\rho - 5\sin\phi a_\phi$ where $\phi = \arctan\frac{4}{3} = 53.13^0$

$$\boxed{Q = 3a_\rho - 4a_\phi}$$

c) $A = 4a_x - 2a_y - 4a_z$.Transforming to cylindrical coordinates the components are

$$A_\rho = 4\cos\phi - 2\sin\phi$$

$$A_\phi = -4\sin\phi - 2\cos\phi$$

$$A_z = -4$$

$$\phi = \arctan\frac{3}{2} = 56.3^0 \quad \cos\phi = 0.55, \quad \sin\phi = 0.832$$

$$A_{cy} = 0.54a_\rho - 4.44a_\phi - 4a_z$$

Problems: Coordinate Transformations

1. Transform the following vector

$$G = \frac{xz}{y} a_x \tag{2.43}$$

   into spherical coordinates.

2. Transform the vector $B = ya_x - xa_y + za_z$ into cylindrical coordinates.

3. Give

   (a) The cartesian coordinates of the point $C(\rho = 4.4, \phi = -115^0, z = 2)$

   (b) The cylindrical coordinates of the point $D(x = -3.1, y = 2.6, z = -3)$

   (c) Specify the distance from C to D

4. Transform to cylindrical coordinates

   (a) $F = 10a_x - 8a_y + 6a_z$, at point $P(10, -8, 6)$

   (b) $G = (2x + y)a_x - (y - 4x)a_y$ at point $Q(\rho, \phi, z)$

   (c) Give the cartesian components of the vector $H = 20a_\rho - 10a_\phi + 3a_z$ at $P(x = 5, y = 2, z = -1)$

5. Given the two points $C(-3, 2, 1)$ and $D(r = 5, \theta = 20^0, \phi = -70^0)$, find

   (a) The spherical coordinates of C

   (b) The cartesian coordinates of D

   (c) the distance from C to D

6. Transform the following vectors to spherical coordinates at the points given
   (a) $10a_x$ at $P(x = 3, y = 2, z = 4)$
   (b) $10a_y$ at $Q(\rho = 5, \phi = 30^0.z = 4)$
   (c) $10a_z$ at $M(r = 4, \theta = 110^0, \phi = 120^0)$
7. Given points $A(\rho = 5, \phi = 70^0, z = -3)$ and $B(\rho = 2, \phi = -30^0, z = 1)$ find
   (a) A unit vector in cartesian coordinates at A directed towards B
   (b) A unit vector in cylindrical coordinates at A directed towards B
   (c) A unit vector in cylindrical coordinates at B directed towards A
8. Express the vector field $D = (x^2 + y^2)^{-1}(xa_x + ya_y)$
   (a) In cylindrical components and cylindrical variables
   (b) Evaluate D at the point where $\rho = 2, \phi = 0.2\pi(rad)$, $z = 5$. Express the result in both cylindrical and cartesian components.
9. Determine an expression for
   (a) a_y in spherical coordinates at $P(r = 0.8, \theta = 30^0, \phi = 45^0)$
   (b) Express a_r in cartesian components at P.
10. Determine the cartesian components of the vector from
   (a) $A(r = 5, \theta = 110^0, \phi = 200^0)$ to $B(r = 7, \theta = 30^0, \phi = 70^0)$

(b) Find the spherical components of the vector at $P(2, -3, 4)$ extending to $Q(-3, 2, 5)$

(c) If $D = 5a_r - 3a_\theta + 4a_\phi$, find $D.a_\rho$ at $M(1, 2, 3)$

2.4 REVIEW OF VECTOR ANALYSIS:

Scalars refer to quantities whose value may be represented by a single real number. Examples are

- Temperature
- Mass
- Density, Pressure
- Volume
- Volume Resistivity
- Voltage

A vector quantity has both a magnitude and a direction in space. Examples are

- Force
- Velocity
- Acceleration
- Electric Field Intensity

2.4.1 VECTOR COMPONENTS AND UNIT VECTORS:

First let us consider Cartesian coordinate system. A vector can be identified by giving the three component vectors, lying along the three coordinate axes whose vector sum must be the given vector. So a vector r can be represented in terms of unit vectors as

$$r = xa_x + ya_y + za_z \tag{2.44}$$

As an example the vector from the origin $(0, 0, 0)$ to a point P$(1, 2, 3)$ is represented as

$$r_P = a_x + 2a_y + 3a_z \tag{2.45}$$

A vector from P$(1, 2, 3)$ to Q$(2, -2, 1)$ is therefore

$$R_{PQ} = r_Q - r_P = (2-1)a_x + (-2-2)a_y + (1-3)a_z = a_x - 4a_y - 2a_z \tag{2.46}$$

The vectors $r_P, r_Q,$ *and* R_{PQ} are shown in figure.

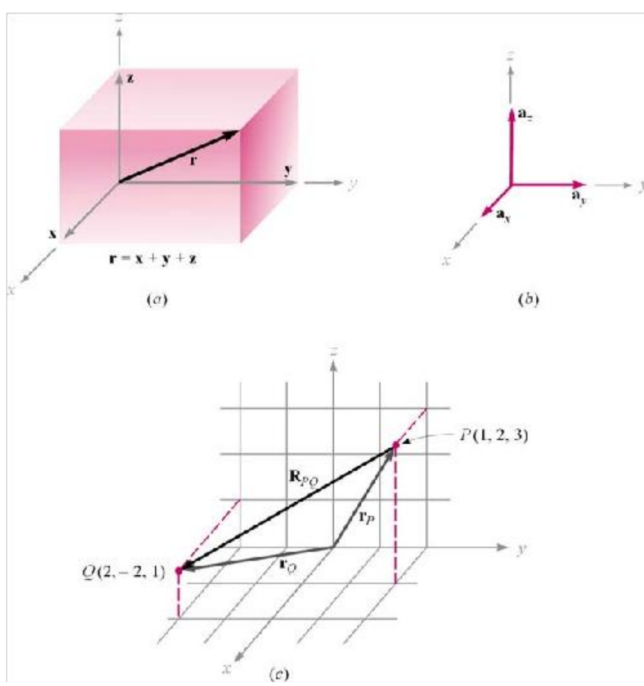


Figure 2.11: Vector components of a vector

Any vector A can be represented by $A = A_x a_x + A_y a_y + A_z a_z$. The magnitude of A is given by

$$|A| = \sqrt{A_x^2 + A_y^2 + A_z^2} \tag{2.47}$$

Each of the three coordinate systems will have its three fundamental and mutually perpendicular unit vectors which are used to resolve any vector into its component vectors.

A unit vector in the direction of A is given by

$$a_A = \frac{A_x a_x + A_y a_y + A_Z a_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}} \tag{2.48}$$

We will use the lower case letter a with an appropriate subscript to designate a unit vector in a specified direction.

2.4.1.1 THE DOT OR SCALAR PRODUCT:

Given two vectors $A \; and \; B$, the dot or scalar product is defined as the product of the magnitude of A , the magnitude of B, and the cosine of the angle between them,

$$A \bullet B = |A|\,|B| \cos\theta_{AB} \tag{2.49}$$

The result is a scalar and also

$$A \bullet B = B \bullet A \tag{2.50}$$

The most important applications of dot product are work done

$$W = \int F \bullet dl \tag{2.51}$$

and calculation of flux ϕ from B the flux density

$$\phi = \iint B \bullet ds \tag{2.52}$$

An expression for the dot product not involving the angle is

$$A \bullet B = A_x B_x + A_y B_y + A_z B_z \tag{2.53}$$

A vector dotted with itself yields the magnitude squared of that particular vector

$$A \bullet A = A^2 = |A|^2 \tag{2.54}$$

Another important application of the dot product is that of finding the component of a vector in a given direction. Refer to fig. . The component of B in the direction specified by the unit vector a is given by

$$B \bullet a = |B|\,|a| \cos\theta_{AB} = |B| \cos\theta_{Ba} \tag{2.55}$$

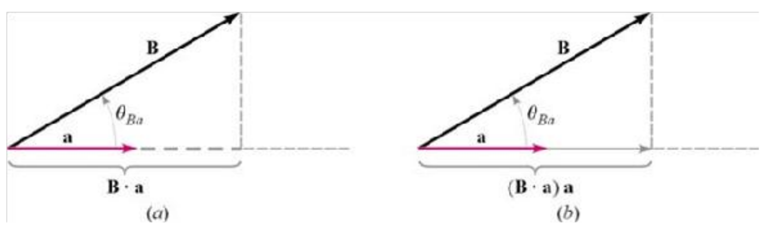


Figure 2.12: Component of vector B in the direction of a

The sign of the component is positive if $0 \leq \theta_{Ba} < 90^0$ and negative whenever $90^0 < \theta_{Ba} \leq 180^0$. In order to obtain the component of a vector B in the direction of a_x we simply take the dot product of B with a_x or $B_x = B \bullet a_x$and the component vector is $B_x a_x$ or $(B \bullet a_x)a_x$ .So the problem of finding the component of a vector in any desired direction boils down to the problem of finding a unit vector in that direction.

The term projection also is used with the dot product . Thus $B \bullet a$ is the projection of B in the direction of a .

2.4.1.2 THE CROSS PRODUCT:

Given two vectors A and B we can define the cross product, or the vector product of A and B as

$$A \times B \tag{2.56}$$

The cross product is a vector. The magnitude of $A \times B$ is equal to the product of the magnitudes of A, B and the sinof the smaller angle between $A\ and\ B$. The direction of $A \times B$ is perpendicular to the plane containing A and B and is along that one of the two possible normals which is in the direction of advance of a right handed screw as A is turned into B through the smaller angle. The direction is illustrated in Fig.

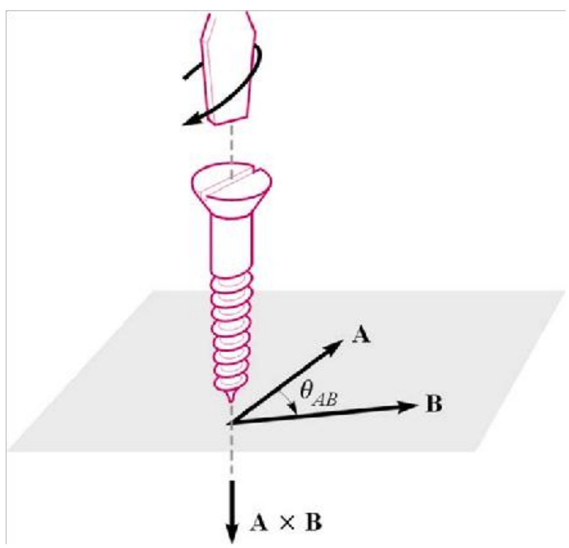


Figure 2.13: Cross product

As an equation

$$A \times B = |A|\,|B| \sin \theta_{AB}\ a_N \tag{2.57}$$

$$A \times B = -(B \times A) \tag{2.58}$$

Also

$$A \times B = (A_y B_z - A_z B_y)a_x + (A_z B_x - A_x B_z)a_y + (A_x B_y - A_y B_x)a_z \tag{2.59}$$

Which can be written as

$$A \times B = \begin{vmatrix} a_x & a_y & a_z \\ A_x & A_y & A_z \\ B_x & B_Y & B_z \end{vmatrix} \tag{2.60}$$

2.4.2 VECTOR CALCULUS, GRADIENT, DIVERGENCE AND CURL:

2.4.2.1 LINE INTEGRALS OF VECTORS:

Certain parameters in electromagnetics are defined in terms of the line integral of a vector field component in the direction of a given path. The component of a vector along a given path is found using the dot product. The resulting scalar function is integrated along the path to obtain the desired result. The line integral of the vector A along the path L is then defined as

$$\int_L A \bullet dl \tag{2.61}$$

see the fig.

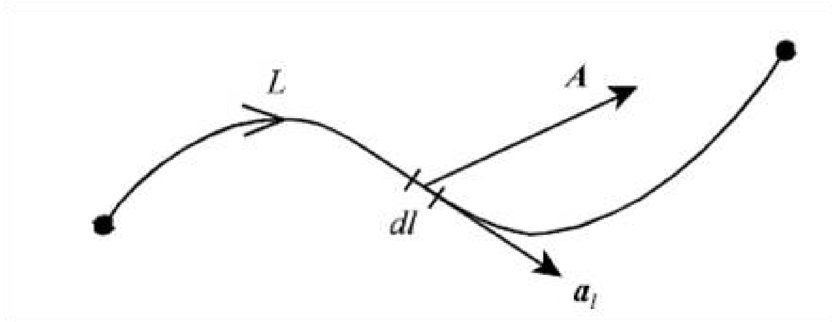


Figure 2.14: Line integral of a vector A

$$
\begin{aligned}
dl &= a_i dl \\
a_l &= \text{Unit vector in the direction of the path L} \\
dl &= \text{Differential element of length alongthe path L} \\
A \bullet dl &= A \bullet a_l dl = A_l dl \\
A_l &= \text{Component of A along the path L}
\end{aligned}
$$

$$\int A \bullet dl = \int_L A_l dl \tag{2.62}$$

whenever the path L is a closed path, the resulting line integral of A is defined as the circulation of A around L and is written as

$$\oint_L A \bullet dl = \oint_L A_l dl \tag{2.63}$$

2.4.2.2 SURFACE INTEGRALS OF VECTORS:

Certain parameters in electromagnetics are defined in terms of the surface integral of a vector field component normal to the surface.The component of a vector normal to the surface is found using the dot product . The resulting scalar function is integrated over the surface to obtain the desired result. The surface integral of the vector A over the surface S (also called the flux of A through S) is then defined as

$$\iint_s A \bullet ds \tag{2.64}$$

see fig.

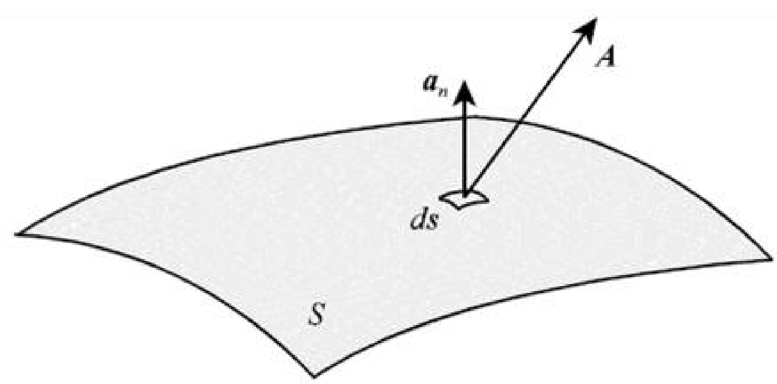


Figure 2.15: Surface Integral of A over S

$$dS = a_n ds$$
$$a_n = \text{Unit vector normal to the suface S}$$
$$dS = \text{Differential surface element on S}$$
$$A \bullet ds = A.a_n ds = A_n ds$$
$$A_n \ \text{ Component of A normal to the surface S}$$

$$\iint_s A \bullet ds = \iint_s A ds \tag{2.65}$$

For a closed surface S, the resulting surface integral of A is defined as the net outward flux of A through S assuming that the unit normal is an outward pointing normal to S

$$\oint_s A \bullet ds = \oint_s A_n ds \tag{2.66}$$

2.4.2.3 THE GRADIENT

A single valued scalar function of the space coordinates x, y, z is denoted by say V. It is a function of position or location only. The points in space at which V has a given value, for example C, define a surface which is referred to as constant value surface. Any number of such surfaces, for various assumed values of the constant C, may be mapped. Such a map shows how the function V varies. The regions where the surfaces are far apart indicate that the functions is slowly varying and if they are closely spaced it indicates that the function is rapidly varying.The rate at which V varies in any given direction at a given point in space is called the directional derivative of V.

It can be seen that the directional derivative of V is a maximum at a given point if the derivative is taken in a direction normal to the constant value surface passing through that point, because the distance between neighboring surfaces is smallest in the normal direction. This maximum value of the directional derivative is called the normal derivative of V.

Let V be a function of rectangular coordinates $V(x, y, z)$. A differential change in this function is given by

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz \tag{2.67}$$

If the differential distance is $dl = dxa_x + dya_y + dza_z$ then

$$dV = G \bullet dl \tag{2.68}$$

where

$$G = \frac{\partial V}{\partial x}a_x + \frac{\partial V}{\partial y}a_y + \frac{\partial V}{\partial z}a_z \tag{2.69}$$

then the incremental change in V can be written as

$$dV = |G|\,|dl|\cos\theta \tag{2.70}$$

where θ is the angle between G and the length vector dl which is along some chosen path. Clearly the maximum space rate of change of V will occur when $\theta = 0$, that is if we move in the direction of G. The direction in which this maximum space rate of change of V takes place is called the gradient of V. Usually the gradient of V is denoted by ∇V. Movement along lines of constant V result in no change in V or $dV = 0$. This shows that $G = \nabla V$ is normal to the constant V surface.

2.4.2.4 PROPERTIES OF GRADIENT OF $V(\nabla V)$:

1. The magnitude of ∇V equals the maximum rate of change of V per unit distance.

2. ∇V points in the direction of maximum rate of change of V

3. ∇V at any point is perpendicular to the constant V surface that passes through that point.

4. The projection or component of ∇V in the direction of a unit vector a is $\nabla V \bullet a$ and is called the directional derivative of V in the direction of a. gradient provides both the direction in which V changes most rapidly and the magnitude of the maximum directional derivative of V

5. If $A = \nabla V$, then V is called the scalar potential of A

2.4.2.5 EXPRESSION FOR GRADIENT IN DIFFERENT CO-ORDINATE SYSTEMS:

$$\text{Cartesian } \nabla \mathrm{V} = \frac{\partial \mathrm{V}}{\partial \mathrm{x}}\mathrm{a_x} + \frac{\partial \mathrm{V}}{\partial \mathrm{y}}\mathrm{a_y} + \frac{\partial \mathrm{V}}{\partial \mathrm{z}}\mathrm{a_z} \tag{2.71}$$

$$\text{Cylindrical } \nabla \mathrm{V} = \frac{\partial \mathrm{V}}{\partial \rho}\mathrm{a}_\rho + \frac{1}{\rho}\frac{\partial \mathrm{V}}{\partial \phi}\mathrm{a}_\phi + \frac{\partial \mathrm{V}}{\partial \mathrm{z}}\mathrm{a_z} \tag{2.72}$$

$$\text{Spherical } \nabla \mathrm{V} = \frac{\partial \mathrm{V}}{\partial \mathrm{r}}\mathrm{a_r} + \frac{1}{\mathrm{r}}\frac{\partial \mathrm{V}}{\partial \theta}\mathrm{a}_\theta + \frac{1}{\mathrm{r}\sin\theta}\frac{\partial \mathrm{V}}{\partial \phi}\mathrm{a}_\phi \tag{2.73}$$

2.4.3 FLUX AND DIVERGENCE OF A VECTOR FIELD:

2.4.3.1 SURFACE INTEGRAL AND FLUX OF A VECTOR FIELD:

A closed surface is a boundary which divides a volume into two parts, an inside and an outside . The surface itself is unbounded. An elemental area is represented by ds, a vector of magnitude $|ds|$ which points in the direction from inside of the volume towards the outside(outward drawn normal).

An open surface is one which is bounded by a curve. The page of a book is an open surface. the magnitude of the area is $|ds|$ and the normal is one of the two normals. The direction which is chosen as positive, is related to the positive sense of traversing the perimeter by the following convention. If a right hand screw is turned in such a direction as to follow in general, the positive sense of the perimeter, then the screw will advance in the direction of the positive normal to the surface. I f the travel is in the counter clockwise direction the normal is up. If clockwise the normal is down.

The flux of a vector field F is defined for an open surface Σ by $\int_\Sigma F \bullet ds$. For a closed surface the flux is defined as $\oint F \bullet ds$

2.4.3.2 THE DIVERGENCE:

The divergence of a vector function F at a point is defined as

$$\nabla \bullet F = \lim_{v \to 0} \left[\frac{1}{v} \oint F \bullet ds \right] \tag{2.74}$$

2.4.3.3 EXPRESSION FOR DIVERGENCE IN CARTESIAN CO-ORDINATES:

Consider a differential cube of volume $dv = dxdydz$ See fig. 2.16

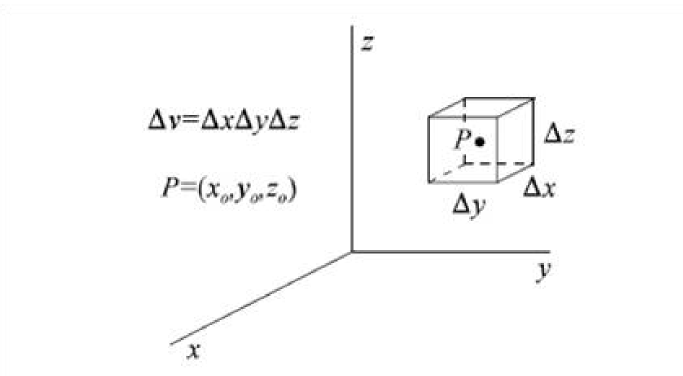


Figure 2.16: Derivation for divergence in Cartesian coordinates

The cube is placed in a vector field D. The total flux passing through the cube can be obtained as flux passing through the front + back face, top + bottom face, side left + side right. For the front face

$$x = x_0 + \frac{dx}{2},\ ds = dydza_x \tag{2.75}$$

$$\int D \bullet ds = \left[D_x(x_0, y_0, z_0) + \frac{dx}{2}\frac{\partial D_x}{\partial x} \right] dydz \tag{2.76}$$

For the back face

$$x = x_0 - \frac{dx}{2},\ ds = dydz(-a_x) \tag{2.77}$$

$$\int D \bullet ds = -\left[D_x(x_0, y_0, z_0) - \frac{dx}{2}\frac{\partial D_x}{\partial x} \right] dydz \tag{2.78}$$

Front +back

$$\frac{\partial D_x}{\partial x} dxdydz \tag{2.79}$$

similarly for the other faces. So the total flux passing through the differential volume dv is

$$\oint D \bullet ds = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) dxdydz \tag{2.80}$$

$$\lim_{dv \to 0} \frac{\oint D \bullet ds}{dv} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \tag{2.81}$$

which is by definition divD. So the expression for divergence in Cartesian coordinate system is

$$\nabla \bullet D = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \tag{2.82}$$

In the other two coordinate systems

$$Cylindrical \ \nabla \bullet D = \left(\frac{1}{\rho} \frac{\partial(\rho D_\rho)}{\partial \phi} + \frac{1}{\rho} \frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z} \right) \quad (2.83)$$

$$Spherical \ \nabla \bullet D = \left(\frac{1}{r^2} \frac{\partial(r^2 D_r)}{\partial r} + \frac{1}{r \sin\theta} \frac{\partial(D_\theta \sin\theta)}{\partial \theta} + \frac{1}{r \sin\theta} \frac{\partial D_\phi}{\partial \phi} \right) \quad (2.84)$$

2.4.3.4 PROPERTIES OF DIVERGENCE:

1. Divergence produces a scalar field from a vector field
2. The divergence of a scalar makes no sense
3. $\nabla \bullet (A + B) = \nabla \bullet A + \nabla \bullet B$
4. If $V$is a scalar $\nabla \bullet (VA) = V\nabla \bullet A + A \bullet \nabla V$

2.4.3.5 GEOMETRICAL INTERPRETATION:

$\nabla \bullet D$ is a measure of how much the vector D spreads out (diverges) from the point question. The vector function A has a large positive divergence at the point if large number of arrows are spreading out.If the arrows point in, it would be a large negative divergence and on the other hand if the lines are parallel and uniform then the divergence is zero.

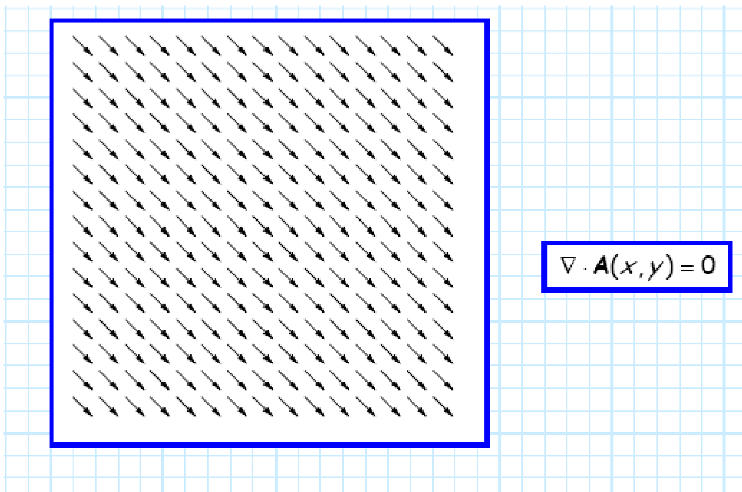


Figure 2.17: The Divergence is zero

The figure below shows two cases where the divergence is negative and where the divergence is positive.

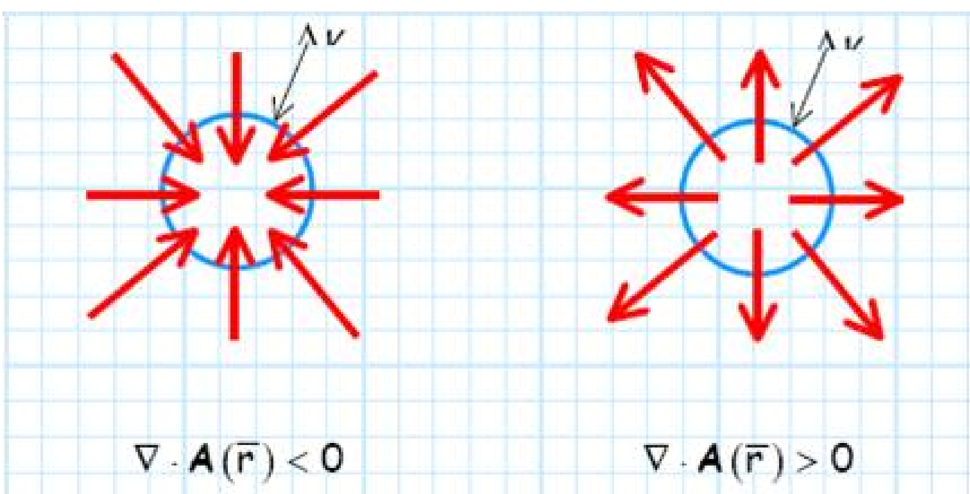


Figure 2.18: Negative and positive divergence

The figure below shows two cases where the divergence is zero.

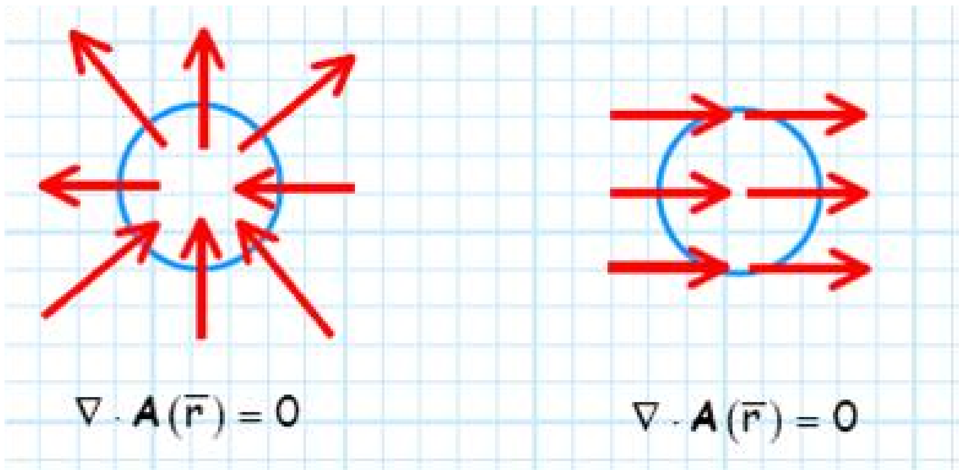


Figure 2.19: Zero Divergence

The figure below shows a field whose value steadily increases as we go away from the y - axis.

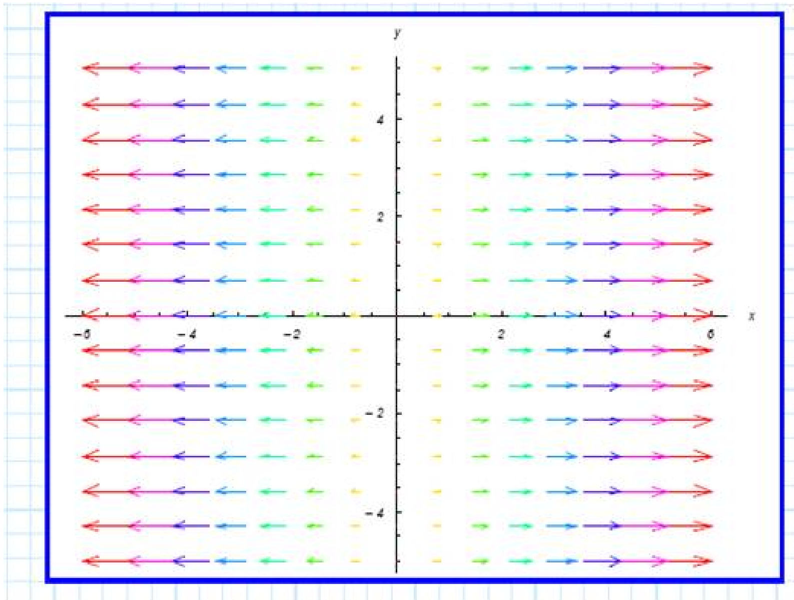


Figure 2.20: Vector field whose value steadily increases as we go away from y axis

The vector field in the above figure is given by $F = |x|\, a_x$ The divergence of which is 1.

2.4.3.6 THE DIVERGENCE THEOREM:

The divergence theorem states that the total outward flux of a vector field A through the closed surface S is the same as the volume integral of the divergence of A. In mathematical form

$$\oint_S A \bullet ds = \int_v (\nabla \bullet A) dv \tag{2.85}$$

2.4.3.7 PROOF OF DIVERGENCE THEOREM:

Consider

$$\oint_{s} A \bullet ds = \sum_{i-1}^{N} \oint_{s_i} A \bullet ds_i = \sum_{i=1}^{N} V_i \left[\frac{\oint_{s_i} A \bullet ds_i}{V_i} \right] \tag{2.86}$$

in the limit $N \to \infty$, $V_i \to 0$, the term in the brackets becomes the divergence of F and the sum goes into volume integral resulting in

$$\oint_{s} A \bullet ds = \int_{V} (\nabla \bullet A) dv \tag{2.87}$$

2.4.3.8 CURL OF A VECTOR AND THE STOKE'S THEOREM:

Circulation of a vector A around a closed path L is the integral $\oint A \bullet dl$. Curl can be defined as an axial vector whose magnitude is the maximum circulation of A per unit area as the area tends to zero and whose direction is the normal to the area, when the area is oriented so as to make the circulation maximum.

$$Curl\, A = \nabla \times A = \left(\lim_{\triangle s \to 0} \frac{\oint A \bullet dl}{\triangle s} \right)_{max} a_n \tag{2.88}$$

where the area $\triangle s$ is bounded by the curve L and a_n is the unit vector normal to the surface $\triangle s$ and is determined using the right hand rule.

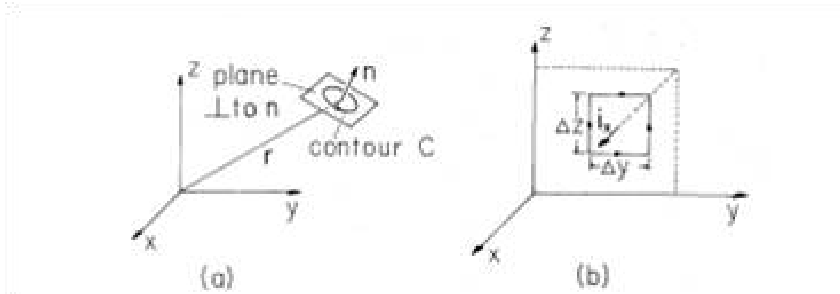


Figure 2.21: Derivation of curl

2.4.3.9 EXPRESSION FOR CURL IN CARTESIAN COORDINATES:

Consider a differential area in the $y-z$ plane. Let the sides of the area element be dy, dz. The closed line integral around the pa

$$\oint A \bullet dl = \left(\int_{ab} + \int_{bc} + \int_{cd} + \int_{da} \right) A \bullet dl, \quad along\, ab \;\; dl = dya_y, \quad z = z_0 - \frac{dz}{2}$$

Let the vector field at the center of the closed loop be $A(x_0, y_0, z_0)$ then

$$\int_{ab} A \bullet dl = \left[A_y(x_0, y_0, z_0) - \frac{dz}{2}\frac{\partial A_y}{\partial z} \right] dy \tag{2.89}$$

similarly

$$\int_{bc} A \bullet dl = \left[A_z(x_0, y_0, z_0) - \frac{dy}{2}\frac{\partial A_z}{\partial y} \right] dy \tag{2.90}$$

$$\int_{cd} A \bullet dl = \left[A_y(x_0, y_0, z_0) + \frac{dz}{2}\frac{\partial A_y}{\partial z} \right] (-dy) \tag{2.91}$$

$$\int\limits_{da} A \bullet dl = \left[A_z(x_0, y_0, z_0) - \frac{dy}{2}\frac{\partial A_z}{\partial y} \right] dy \tag{2.92}$$

Let $\triangle s = dydz$ then adding all the four integrals we get the $x - component$ of the curl.

$$\lim_{\triangle s \to 0} \oint \frac{A \bullet dl}{\triangle s} = (curl)_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \tag{2.93}$$

similarly

$$(curl)_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \tag{2.94}$$

$$(curl)_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \tag{2.95}$$

then

$$Curl\ A = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) a_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) a_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) a_z \tag{2.96}$$

This can also be written as

$$Curl\, A = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \tag{2.97}$$

2.4.3.10 STOKE'S THEOREM:

Stoke's theorem states that the circulation of a vector field A around a closed path L is equal to the surface integral of the curl of A over the open surface S bounded by L provided that A

and $\nabla \times A$ are continuous on S .in mathematical terms it can be written as

$$\oint_{c} A \bullet dl = \iint (\nabla \times A) \bullet ds \tag{2.98}$$

2.4.3.11 PROOF OF STOKE'S THEOREM:

Consider

$$\oint_{c} A \bullet dl = \sum_{i=1}^{N} \oint_{c} A \bullet dl_i = \sum_{i=1}^{N} ds_i \left(\frac{\oint_c A \bullet dl_i}{ds_i} \right) \tag{2.99}$$

Observe what happens to the right hand side as N is made enormous and ds_i shrink. The quantity in the parentheses becomes $(\nabla \times A) \bullet a_i$ where a_i is the unit vector normal to the i th patch.. So we have on the right the sum, over all the patches that make up the entire surface S spanning C, of the product "patch area times normal component of (Curl of A)". This is nothing but the surface integral over S , of the vector curl A

$$\sum_{i=1}^{N} ds_i \left(\frac{\oint_c A \bullet dl_i}{ds_i} \right) = \sum_{i=1}^{N} ds_i (\nabla \times A) \bullet a_i = \int_{s} (\nabla \times A) \bullet ds \tag{2.100}$$

$$\boxed{\oint_c A \bullet dl = \int_s (\nabla \times A) \bullet ds}$$

It relates the line integral of a vector to the surface integral of the curl of the vector.

2.4.3.12 PROPERTIES OF CURL:

1. The curl of a vector field is another vector field
2. $\nabla \times (A + B) = \nabla \times A + \nabla \times B$
3. $\nabla \times (A \times B) = A(\nabla \bullet B) - B(\nabla \bullet A) + (B \bullet \nabla)A - (A \bullet \nabla)B$
4. The divergence of the curl of a vector field is zero
5. The curl of the gradient of a scalar is zero

(2.101)

2.4.3.13 CLASSIFICATION OF VECTOR FIELDS:

All fields can be classified in terms of their vanishing or non-vanishing divergence or curl.

$$\begin{array}{ll} \nabla \bullet A = 0, \ \ \nabla \times A = 0 & \nabla \bullet A \neq 0, \ \ \nabla \times A = 0 \\ \nabla \bullet A = 0, \ \ \nabla \times A \neq 0 & \nabla \bullet A \neq 0, \ \ \nabla \times A \neq 0 \end{array}$$

Below are the examples of the fields

1. $A = ka_x$, $\nabla \bullet A = 0, \nabla \times A = 0$. Solenoidal and irrational
2. $A = kr$, $\nabla \bullet A = 3k, \nabla \times A = 0$. Non-solenoidal and irrational.
3. $A = k \times r$, $\nabla \bullet A = 0, \nabla \times A = 2k$. Solenoidal and rotational.
4. $A = k \times r + cr$,$\nabla \bullet A = 3c, \nabla \times A = 2k$. Non-solenoidal and rotational.

A vector field A is said to be solenoidal (divergence less) if $\nabla \bullet A = 0$. Such a field has neither a source nor a sink of flux.

A vector field is said to be irrational if $\nabla \times A = 0$

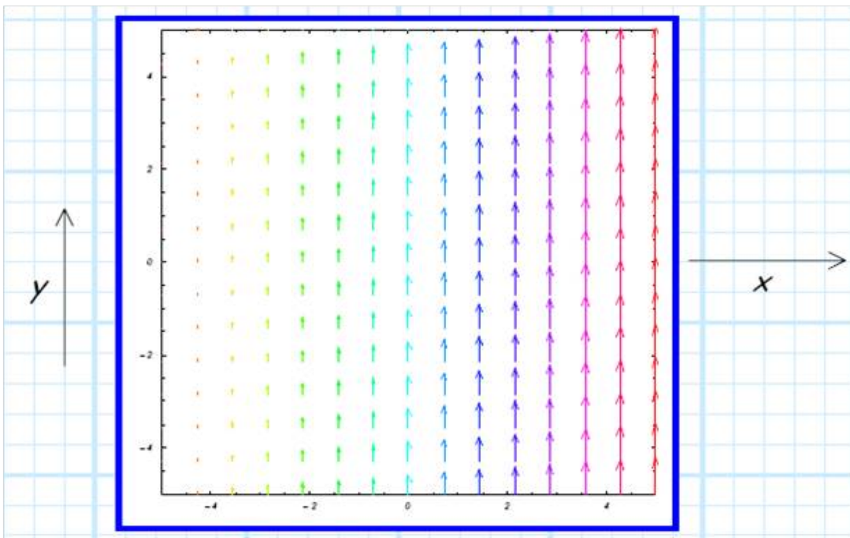


Figure 2.22:

2.4.3.14 HELMHOLTZ'S THEOREM:

To what extent is a vector function determined by its divergence and curl? Suppose we are told that the divergence of $F$is a specified scalar function D

$$\nabla \bullet F = D \tag{2.102}$$

and the curl of F is a specified function C

$$\nabla \times F = C \tag{2.103}$$

(for consistency, C must be divergence less $\nabla \bullet C = 0$ because the divergence of a curl is always zero). On the basis of this information, can the function F be found? If this information is not sufficient, there may be more than one solution to the problem; if there is too much of information, there may not be any solution. Helmholtz's theorem provides the answer to this:

HELMHOLTZ'S THEOREM: If the divergence $D(r)$ and the curl $C(r)$ of a vector function $F(r)$ are specified, and if they both go to zero faster than $\frac{1}{r^2}$ as $r \to \infty$ and if $F(r)$ goes to zero as $r \to \infty$, then F is given uniquely by

$$F = -\nabla U + \nabla \times W \tag{2.104}$$

where U is a scalar field and W is a vector field.

Corollary: Any vector function $F(r)$, which goes to zero faster than $\frac{1}{r}$ as $r \to \infty$, can be expressed as the gradient of a scalar plus the curl of a vector:

$$F = \nabla\left(-\frac{1}{4\pi}\int\frac{\nabla \bullet F}{r}d\tau\right) + \nabla \times \left(\frac{1}{4\pi}\int\frac{\nabla \times F}{r}d\tau\right) \tag{2.105}$$

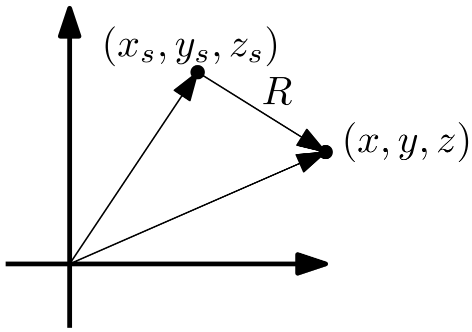
2.4.3.15 VECTOR IDENTITIES:

1. $\nabla(U+V) = \nabla U + \nabla V$
2. $\nabla(UV) = U\nabla V + V\nabla U$
3. $\nabla\left(\frac{U}{V}\right) = \frac{V(\nabla U) - U(\nabla V)}{V^2}$
4. $\nabla V^n = nV^{n-1}\nabla V$ (n = integer)
5. $\nabla(A \bullet B) = (A\bullet\nabla)B + (B\bullet\nabla)A + A\times(\nabla\times B) + B\times(\nabla\times A)$
6. $\nabla \bullet (A+B) = \nabla \bullet A + \nabla \bullet B$

7. $\nabla \bullet (A \times B) = B \bullet (\nabla \times A) - A \bullet (\nabla \times B)$
8. $\nabla \bullet (VA) = V\nabla \bullet A + A \bullet \nabla V$ where V is a scalar
9. $\nabla \bullet (\nabla V) = \nabla^2 V$
10. $\nabla \bullet (\nabla \times A) = 0$
11. $\nabla \times (A + B) = \nabla \times A + \nabla \times B$
12. $\nabla \times (A \times B) = A(\nabla \bullet B) - B(\nabla \bullet A) + (B \bullet \nabla)A - (A \bullet \nabla)B$
13. $\nabla \times (VA) = \nabla V \times A + V(\nabla \times A)$
14. $\nabla \times (\nabla V) = 0$
15. $\nabla \times (\nabla \times A) = \nabla(\nabla \bullet A) - \nabla^2 A$
16. $\oint_L A \bullet dl = \int_s (\nabla \times A) \bullet ds$
17. $\oint_L V dl = -\int_s \nabla \times ds$
18. $\oint_s A \bullet ds = \int_v (\nabla \bullet A) dv$
19. $\oint_s V ds = \int_v \nabla V dv$
20. $\oint_s A \times ds = -\int_v \nabla \times A dv$

Tutorial and Homework problems

1. What is the physical definition of the gradient of scalar fields?
2. Express the space rate of change of a scalar in a given direction in terms of its gradient.
3. What is the physical definition of the divergence of a vector field?
4. What is the physical definition of the curl of a vector field?
5. What is the difference between an irrotational field and a solenoidal field?
6. Given a vector field $F = ya_x + xa_y$, evaluate the integral $\int F \bullet dl$ from $P_1(2, 1, -1)$ to $P_2(8, 2, -1)$
   (a) along the straight line joining the two points, and
   (b) along the parabola $x = 2y^2$. Is this F a conservative field.
7. Given a vector field $F = xya_x + yza_y + zxa_z$
   (a) Compute the total outward flux from the surface of a unit cube in the first octant with one corner at the origin.
   (b) Find $\nabla \bullet F$ and verify the divergence theorem.
8. Obtain $\nabla(\frac{1}{R})$, considering the point (x_s, y_s, z_s) in the figure below as fixed while the point (x, y, z) as variable.



9. Obtain $\nabla_s \left(\frac{1}{R}\right)$ for the previous example.

10. Assume that a vector field is given by $A = (2x^2 + y^2)a_x + (xy - y^2)a_y$

    (a) Find $\oint A \bullet dl$ arround the triangular contour shown in the figure below

    (b) Find $\oint (\nabla \times A) \bullet ds$ over the triangular area .

    (c) Can A be expressed as the gradient of a scalar ? Explain.

Unit-I

Electrostatics:

Electrostatic Fields – Coulomb's Law – Electric Field Intensity (EFI) – EFI due to a line and a surface charge – Work done in moving a point charge in an electrostatic field – Electric Potential – Properties of potential function – Potential gradient – Gauss's law, Application of Gauss's Law – Maxwell's first law, $\nabla \bullet D = \rho_v$
.

Chapter 3

STATIC ELECTRIC FIELDS

Learning Outcomes

- Define electric charge, and describe how the two types of charge interact.
- Describe three common situations that generate static electricity.
- State the law of conservation of charge.
- State Coulomb's law in terms of how the electrostatic force changes with the distance between two objects.
- Calculate the electrostatic force between two point charges.
- Compare the electrostatic force to the gravitational force

Electrostatics is the study of the effects of electric charges at rest, and the electric fields do not change with time. Although this is the simplest situation in electromagnetics, its mastery is fundamental to the understanding of more complicated electromagnetic models. The explanation of many natural phenomena (such as lightning and corona) and principles of some important industrial applications (such as oscilloscopes,ink-jet printers, xerography, capacitance key board and liquid crystal displays). are based on electrostatics.

What makes plastic wrap cling? Static electricity. Not only are applications of static electricity common these days, its existence has been known since ancient times. The first record of its effects dates to ancient Greeks who noted more than 500 years B.C. that polishing amber temporarily enabled it to attract bits of straw. The very word electric derives from the Greek word for amber (electron). Many of the characteristics of static electricity can be explored by rubbing things together. Rubbing creates the spark you get from walking across a wool carpet, for example. Static cling generated in a clothes dryer and the attraction of straw to recently polished amber also result from rubbing. Similarly, lightning results from air movements under certain weather conditions. You can also rub a balloon on your hair, and the static electricity created can then make the balloon cling to a wall. We also have to be cautious of static electricity, especially in dry climates. When we pump gasoline, we are warned to discharge ourselves (after sliding across the seat) on a metal surface before grabbing the gas nozzle. Attendants in hospital operating rooms must wear booties with aluminum foil on the bottoms to avoid creating sparks which may ignite the oxygen being used. Some of the most basic characteristics of static electricity include:

- The effects of static electricity are explained by a physical quantity not previously introduced, called electric charge.
- There are only two types of charge, one called positive and the other called negative.
- Like charges repel, whereas unlike charges attract.

- The force between charges decreases with increasing distance.

How do we know there are two types of electric charge? When various materials are rubbed together in controlled ways, certain combinations of materials always produce one type of charge on one material and the opposite type on the other. By *convention,* we call one type of charge "positive", and the other type "negative." For example, when glass is rubbed with silk, the glass becomes positively charged and the silk negatively charged. Since the glass and silk have opposite charges, they attract one another like clothes that have rubbed together in a dryer. Two glass rods rubbed with silk in this manner will repel one another, since each rod has positive charge on it. Similarly, two silk cloths so rubbed will repel, since both cloths have negative charge.

> With the exception of exotic, short-lived particles, all charge in nature is carried by electrons and protons. Electrons carry the charge we have named negative. Protons carry an equal-magnitude charge that we call positive. Electron and proton charges are considered fundamental building blocks, since all other charges are integral multiples of those carried by electrons and protons. Electrons and protons are also two of the three fundamental building blocks of ordinary matter. The neutron is the third and has zero total charge.

Charge has two important properties

1. Charge is quantized
2. Charge is conserved

Quantization of charge means charge is available in nature as integral multiples of the charge of an electron. We can not have $\frac{1}{2}$

charge of an electron or 0.75 times the charge of an electron.

Charge is conserved. It can not be created or destriyed. The total charge of the universe is fixed for all the time.

> Only a limited number of physical quantities are universally conserved. Charge is one—energy, momentum, and angular momentum are others. Because they are conserved, these physical quantities are used to explain more phenomena and form more connections than other, less basic quantities. We find that conserved quantities give us great insight into the rules followed by nature and hints to the organization of nature. Discoveries of conservation laws have led to further discoveries, such as the weak nuclear force and the quark substructure of protons and other particles.

3.1 COULOMB'S LAW

Charles-Augustin de Coulomb: *(born June 14, 1736, Angoulême, France—died August 23, 1806, Paris), French physicist best known for the formulation of Coulomb's law.*
Coulomb spent nine years in the West Indies as a military engineer and returned to France with impaired health. Upon the outbreak of the French Revolution, he retired to a small estate at Blois and devoted himself to scientific research. In 1802 he was appointed an inspector of public instruction. Coulomb developed his law as an outgrowth of his attempt to investigate the law of electrical repulsions as stated by Joseph Priestley of England. To this end he invented sensitive apparatus to measure the electrical forces involved in Priestley's law and published his findings in 1785–89. He also established the inverse square law of attraction and repulsion of unlike and like magnetic poles, which became the basis for the mathematical theory of magnetic forces developed by Siméon-Denis Poisson. He also did research on friction of machinery, on windmills, and on the elasticity of metal and silk fibres. The coulomb, a unit of electric charge, was named in his honour.

3.1.1 FORCE BETWEEN POINT CHARGES:

3.1.1.1 Elecric charge:

The concept of electric charge is fundamental to all electromagnetic phenomena, including electronics, optics, friction, chemistry, etc., but we have noidea what it is! We know what it does , and how big it is, but the fundamental nature of charge is unknown. We have to simply accept that charge exists and that some fundamental particles ,electrons and positrons, have it and others like neutrons , do not.

What we know is that there are two types of charge that we call positive and negative. these are of course arbitrarily chosen names and without any deep significance. We know that electron possesses negative charge and we call the value of the charge as elementary charge. All electrons have the same amount of charge . No exceptions!

The value of the elementary charge is

$$e = 1.602176462 \pm 0.000000063.10^{-19}C \tag{3.1}$$

where the uncetanity is the standard deviation.

Units:
The SI unit for electrical charge is Coulomb, for which we use the symbol C. The magnitude of C is based on magnetic measurements.

One way to illustrate the mysterious ways of charge is to consider the charge of the electron . We know that the radius of the electron must be less than 10^{-17}cm . We can calculate the charge density of the electron as

$$\rho_e = \frac{e}{\frac{4}{3}r_e^3} > 10^{31}\frac{C}{cm^3} \tag{3.2}$$

This is an enormous number that we can not begin to create in any macroscopic object.

If we tried to charge a macroscopic sphere up till its charge density matched this value, the sphere would blow apart long before we succeeded in reaching the electrron's charge density, no matter what type of material is used to construct the sphere!

Coulomb's law is formulated in 1785.It deals with the force a point charge exerts on another point charge. By a point charge we mean a charge that is located on a body whose dimensions are much smaller than other relevant dimensions.

The force between two point charges Q_1 and Q_2 is

1. Along the line joining them
2. Directly proportional to the product of the magnitudes of the charges Q_1 and Q_2 .
3. Inversely proportional to the square of the the distance 'R' , between the charges
4. Like charges repel and unlike charges attract.

Expressed in mathematical form

$$F = k\frac{Q_1 Q_2}{R^2} \tag{3.3}$$

'k' is the proportionality constant

Q_1 and Q_2 are point charges and in Coulombs

'R' is the distance in meters

F is the force in Newtons

In SI system of units $k = \frac{1}{4\pi\epsilon_0}$ where ϵ_0 is the permittivity or dielectric constant of the free space.

$$\epsilon = 8.854 \times 10^{-12} \approx \frac{10^{-9}}{36 \times \pi} \tag{3.4}$$

3.1.2 COULOMB'S LAW IN VECTOR FORM:

Q_1 and Q_2 are located at points '1' an '2' having position vectors r_1 and r_2, then the vector force F_2 on Q_2 due to Q_1 is given by

$$F_{12} = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2} \times a_{R_{12}} \tag{3.5}$$

See Fig3.1.
where

$$R_{12} = r_2 - r_1 \tag{3.6}$$

and

$$a_{R_{12}} = \frac{R_{12}}{|R_{12}|} \tag{3.7}$$

is the unit vector in the direction of the force.

$$F_{12} = \frac{Q_1 Q_2}{4\pi\epsilon_0} \frac{(r_2 - r_1)}{|r_2 - r_1|^3} \tag{3.8}$$

$F_{12} = -F_{21}$

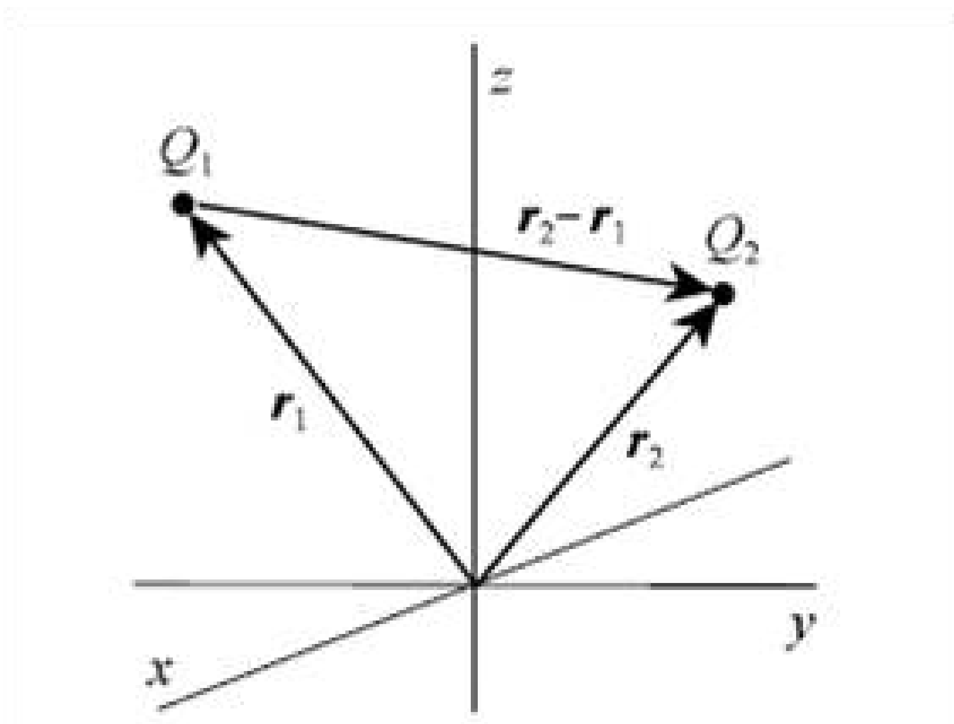


Figure 3.1: Force between two point charges

In formulating this law no hypothesis is made concerning the mechanism by which the force is transmitted over the intervening distance in the vacuum. Either the force is transmitted instantaneously, ie., with infinite speed, or it may be postulated that the speed of transmission of the force is finite , but that all transient effects have disappeared leaving the steady state condition, the one of interest. Either way the situation being considered is a static one.
A comparison of the relative magnitudes of the electrical and gravitational forces between two electrons shows how large are the electrical forces compared to gravitational forces . An electron has the smallest quantum of charge and also the smallest known finite mass : $1.6 \times 10^{-19}C$ *and* $9.1 \times 10^{-31}kg$ *. For two electrons separated by a distance of* $1m$

$$F_{elec} = \frac{1}{4\pi\epsilon_0}\frac{q_s q_t}{R^2} = 9 \times 10^9 \frac{(1.6 \times 10^{-19})^2}{(10^{-3})^2} = 2.3 \times 10^{-22} N$$

$$F_{grav} = G\frac{m_1 m_2}{R^2} = 6.67 \times 1 -^{-11} \frac{(9.1 \times 10^{-31})^2}{(10^{-3})^2} = 5.5 \times 10^{-65} N$$

For electrons the electrical force is almost 10^{43} *as strong as the gravitational force. For other charged particels also it is not different. As a consequence,it is unnecessary to consider the gravitational force when electrical forces are present.*

The Coulomb expression yields an infinite force when two point charges (Finite charge, infinitesimal size, infinite charge density) are separated by an infinitesimal distance. But when one of the charges is itself an infinitesimal, $\rho d\tau_s$ then the force it produces on a point test charge located there (at the same point) is finite.

3.1.3 PRINCIPLE OF SUPERPOSITION:

If there are more than two point charges the principle of super position can be applied to determine the force on a particular charge because of all the reaming charges. The principle states that if there are $'N'$ charges $Q_1, Q_2, Q_3 \ldots Q_N$ located respectively at points whose position vectors are $r_1, r_2, r_3 \ldots r_N$, the resultant force Fon charge Q located at a point whose position vector is r is the vector sum of the forces exerted on Q by charges $Q_1, Q_2, Q_3, \ldots Q_N$ is

$$F = \frac{QQ_1}{4\pi\epsilon_0}\frac{r - r_1}{|r - r_1|^3} + \frac{QQ_2}{4\pi\epsilon_0}\frac{r - r_2}{|r - r_2|^3} + \ldots + \frac{QQ_N}{4\pi\epsilon_0}\frac{r - r_N}{|r - r_N|^3} \quad (3.9)$$

The above can also be expressed as a summation

$$F = \frac{Q}{4\pi\epsilon_0}\sum_{k=1}^{N} Q_k \frac{(r - r_k)}{|r - r_k|^3} \quad (3.10)$$

If $Q_1 = Q_2 = 1$C and $R_{12} = 1$m the force acting between these charges is $= 9 \times 10^9$ N. .An enormous force.

The electrical forces huge

The exponent in Coulomb's law differs from '2' by one part in one billion. Coulomb's law is valid for distances of the order of 10^{-13}*cm. The law fails at distances of the order of* 10^{-14}*cm. It is also valid for distances of several kilo metes.*

Example:

As an example consider a charge of $3 \times 10^{-4}C$ at $P(1, 2, 3)$ and a charge of $-10^{-4}C$ at $Q(2, 0, 5)$ in vacuum. Find the force acting on charge at Q .

Ans:
$Q_1 = 3 \times 10^{-4}$ and $Q_2 = -10^{-4}$

$$R_{12} = r_2 - r_1 = (2-1)a_x + (0-2)a_y + (5-3)a_z \tag{3.11}$$

$$= a_x - 2a_y + 2a_z \tag{3.12}$$

$$\begin{aligned} a_{12} &= \frac{a_x - 2a_y + 2a_z}{3} \\ F_2 &= \frac{3 \times 10^{-4(-10^{-4})}}{4\pi\left(\frac{1}{36\pi}\right)10^{-9}}\left(\frac{a_x - 2a_y + 2a_z}{3}\right) \\ F_2 &= -30\left(\frac{a_x - 2a_y + 2a_z}{3}\right)N \end{aligned}$$

Example:
Point charges $1mC$ and $-2mC$ are located at $(3, 2, -1)$ and $(-1, -1, 4)$ respectively. Calculate the electrical force on a $10nC$ charge located at $(0, 3, 1)$.
Ans:

$$\begin{aligned} F &= \sum_{k=1}^{2} \frac{QQ_k}{4\pi\epsilon_0}\frac{r - r_k}{|r - r_k|^3} \\ F &= 10 \times 10^{-9} \times 9 \times 10^{9} \times 10^{-3}\left(\frac{(-3,1,2)}{[(0,3,1)-(3,2,-1)]^3} - \frac{2(1,4,-3)}{[(0,3,1)-(-1,-1,4)]^3}\right) \\ F &= 9 \times 10^{-2}\left(\frac{(-3,1,2)}{14\sqrt{14}} + \frac{(-2,-8,6)}{26\sqrt{26}}\right) \\ F &= -6.507a_x - 3.817a_y + 7.506a_z mN \end{aligned}$$

Example:

$2mC$ charge (positive) is located at $P_1(3, -2, -4)$ and a $5\mu C$ charge (negative) is at $P_2(1, -4, 2)$

1. Find the vector force on the negative charge
2. Also find the magnitude of the force

Ans:

$$\begin{aligned}
R_{12} &= [(1, -4, 2) - (3, -2, -4)] = -2a_x - 2a_y + 6a_z \\
|R_{12}| &= \sqrt{44}\ , a_{R_{12}} = \frac{-2a_x - 2a_y + 6a_z}{\sqrt{44}} \\
F_{12} &= (2 \times 10^{-3})(-5 \times 10^{-6}) \times 9 \times 10^9 \left(\frac{-2a_x - 2a_y + 6a_z}{44\sqrt{44}} \right) \\
F_{12} &= 0.613a_x + 0.613a_y - 1.84a_z\ N \\
|F_{12}| &= \sqrt{(0.613)^2 + (0.613)^2 + (1.84)^2} = 2.034\ \ N
\end{aligned}$$

Example:

It is required to hold four equal point charges q C each in equilibrium at the corners of a square of side a meters. Prove that the point charge which can do this is a negative charge of magnitude

$$\frac{(2\sqrt{2} + 1)}{4} q \tag{3.13}$$

coulombs placed at the center of the square.

3.2 ELECTRIC FIELD

Consider one charge fixed in position, say Q_1 with position vector R_1 and move a second charge slowly around, it can be seen that there exists everywhere a force on the second charge. In other words, the second charge is displaying the existence of a force field. If the test charge is denoted by Q_t , the force on it is given by Coulomb's law as

$$F_t = \frac{Q_1 Q_t}{4\pi\epsilon_0 R_{1t}^2} a_{R_{1t}} \tag{3.14}$$

Writing this force as a force per unit charge gives

$$\frac{F_t}{Q_t} = \frac{Q_1}{4\pi\epsilon_0 R_{1t}^2} a_{R_{1t}} \tag{3.15}$$

The force is only a function of Q_1 and is a directed segment from Q_1 to the position of the test charge. This is a vector field and is called the **Electric Field intensity.**

Electric field intensity can be defined as vector force on a unit positive test charge. The units are Newton/Coulomb. Anticipating a new quantity Volt which will be defined later, the unit for electric field intensity is normally given by Volt/meter.

The test charge should be small such that it will not disturb the original field of the charge distribution under consideration. So the electric field intensity denoted by E is defined as

$$\lim_{Q_t \to 0} \frac{F_t}{Q_t} \tag{3.16}$$

Electrical field of a point charge

Let the point charge be located at some point in a spherical coordinate system at a distance$r$ from the origin of the co-ordinate system. 'E' can be expressed as

$$E = \frac{Q}{4\pi\epsilon_0 r^2} a_r \tag{3.17}$$

In Cartesian co-ordinates $r = xa_x + ya_y + za_z$ and

$$a_r = \frac{xa_x + ya_y + za_z}{\sqrt{x^2 + y^2 + z^2}} \tag{3.18}$$

The field is spherically symmetric. If the charge is not at the origin, the field will not be spherically symmetric. If the source charge is at $r' = x'a_x + y'a_y + z'a_z$, the field at a general point $r = xa_x + ya_y + za_z$ is given by

$$E = \frac{Q}{4\pi\epsilon_0 \left|r - r'\right|^2} \frac{r - r'}{\left|r - r'\right|} \tag{3.19}$$

3.2.1 ELECTRIC FIELD BECAUSE OF CHARGE DISTRIBUTIONS:

Charge distributions can be of three types

1. Line charge distribution C/m
2. Surface charge distribution C/m^2
3. Volume charge distribution C/m^3 The total charge in a given configuration can be obtained as

$$dQ = \rho_L dl \ \ and \ Q = \int \rho_L dl \tag{3.20}$$

$$dQ = \rho_s ds \ and \ Q = \int \rho_s ds \tag{3.21}$$

$$dQ = \rho_v dv \text{ and } Q = \int \rho_v dv \tag{3.22}$$

The electric fields because of the distributions are given by

$$E = \int \frac{\rho_L dl}{4\pi\epsilon_0 R^2} a_R \tag{3.23}$$

$$\begin{aligned} E &= \int \frac{\rho_s ds}{4\pi\epsilon_0 R^2} a_R \\ E &= \int \frac{\rho_v dv}{4\pi R^2} a_R \end{aligned}$$

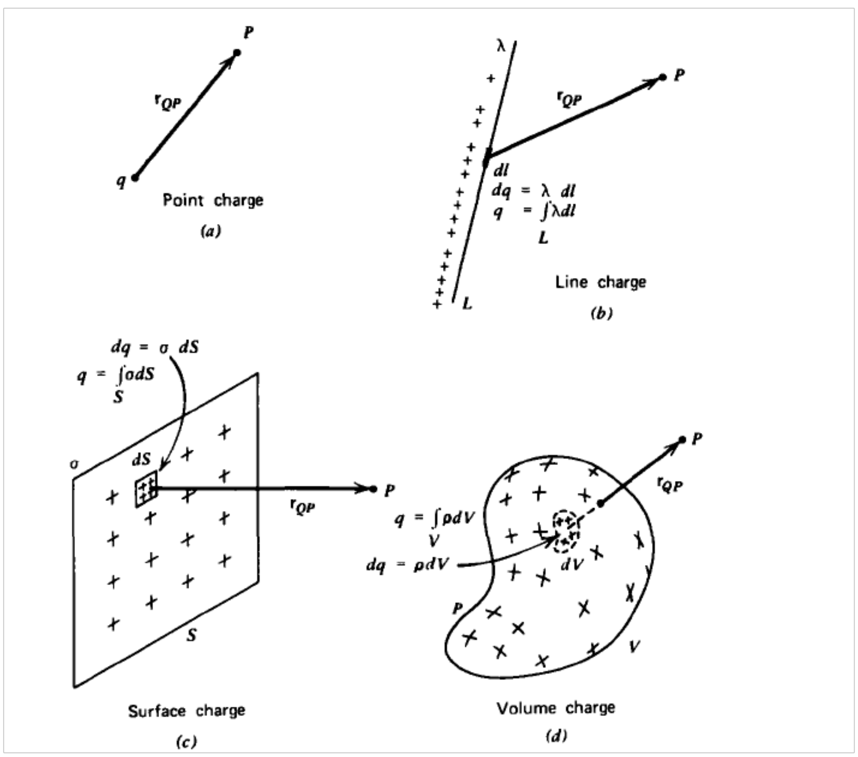


Figure 3.2: Charge Distributions

3.2.2 FIELD BECAUSE OF A FINITE LINE CHARGE:

Consider a line of finite length with uniform charge density ρ_L C/m extending from A to b along the $Z - axis$ as shown in the figure.The charge in the element $dl = dz$ is dQ and is given by $\rho_L dz$. The total charge is given by

$$Q = \int_{z_A}^{z_B} \rho_L dz \tag{3.24}$$

The source point is (x, y, z), the field point is (x', y', z'), so $dl = dz'$

$$R = xa_x + ya_y + (z - z')a_z \tag{3.25}$$

$$R = \rho a_\rho + (z - z')a_z \tag{3.26}$$

$$|R|^2 = \rho^2 + (z - z')^2 \tag{3.27}$$

$$\frac{a_R}{|R|^3} = \frac{[\rho a_\rho + (z - z')a_z]}{[\rho^2 + (z - z')^2]^{\frac{3}{2}}} \tag{3.28}$$

$$Then \tag{3.29}$$

$$E = \frac{\rho_L}{4\pi\epsilon_0} \int \frac{[\rho a_\rho + (z - z')a_z]}{[\rho^2 + (z - z')^2]^{\frac{3}{2}}} \tag{3.30}$$

Define α, α_1 *and* α_2 as shown in the figure.

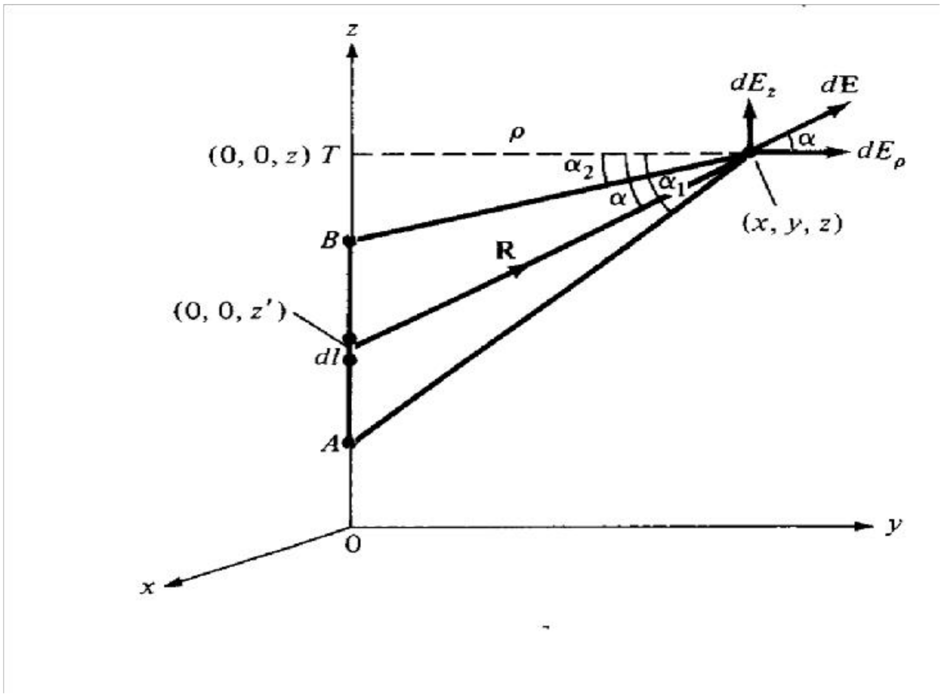


Figure 3.3: Line charge distribution

$$R = [\rho^2 + (z - z')^2]^{\frac{1}{2}} = \rho \sec \alpha \tag{3.31}$$

$$z' = oT - \rho \tan \alpha,\ dz' = -\rho \sec^2 \alpha\ d\alpha \tag{3.32}$$

$$E = -\frac{\rho_L}{4\pi\epsilon_0} \int_{\alpha_1}^{\alpha_2} \frac{\rho \sec^{\alpha} [\cos \alpha a_\rho + \sin \alpha a_z]\, d\alpha}{\rho^2 \sec^2 \alpha} \tag{3.33}$$

$$\cos \alpha = \frac{\rho}{R}, \sin \alpha = \frac{(z - z')}{R} \tag{3.34}$$

$$E = -\frac{\rho_L}{4\pi\epsilon_0}\int_{\alpha_1}^{\alpha_2}[\cos\alpha a_\rho + \sin\alpha a_z]\,d\alpha \qquad (3.35)$$

$$E = \frac{\rho_L}{4\pi\epsilon_0}[(\sin\alpha_1 - \sin\alpha_2)a_\rho + (\cos\alpha_2 - \cos\alpha_1)a_z] \qquad (3.36)$$

3.2.3 CASE I: INFINITE LINE CHARGE:

For an infinite line charge point B will be at $(0, 0, \infty)$ and point A will be at $(0, 0, -\infty)$. so that $\alpha_1 = \frac{\pi}{2}, \alpha_2 = -\frac{\pi}{2}$. Substitution of the above values gives

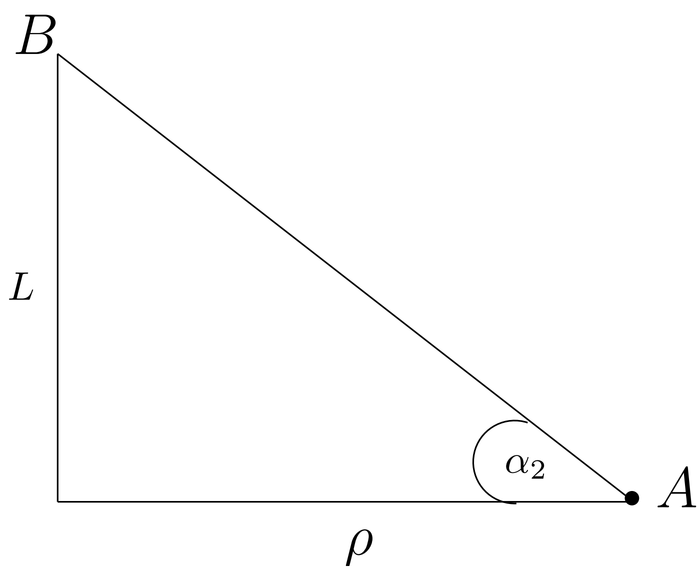
$$E = \frac{\rho_L}{2\pi\epsilon_0\rho}a_\rho \qquad (3.37)$$

The z-component vanishes.

3.2.4 CASE II:LOWER END COINCIDING WITH THE FIELD POINT:

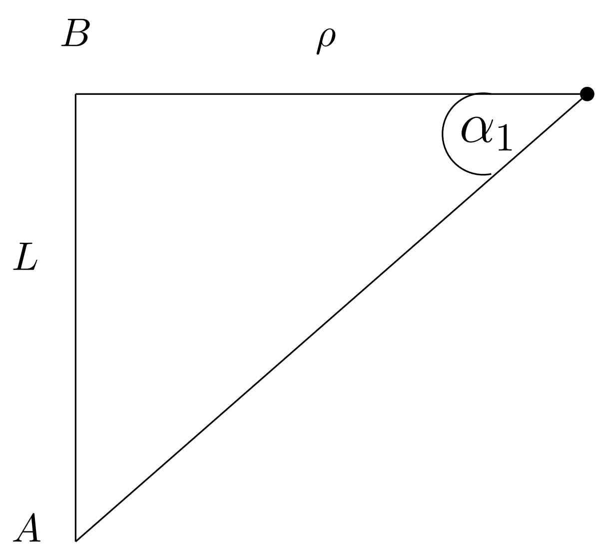
$\alpha_1 = 0,\ \alpha_2 = \tan^{-1}\frac{L}{\rho}$

$$E = \frac{\rho_L}{4\pi\epsilon_0\rho}[(\cos\alpha_2 - 1)a_z + \sin\alpha_2 a_\rho] \qquad (3.38)$$



CaseIII: Upper end coinciding with the field point:
$\alpha_1 = \tan^{-1}\frac{L}{\rho},\ \alpha_2 = 0$

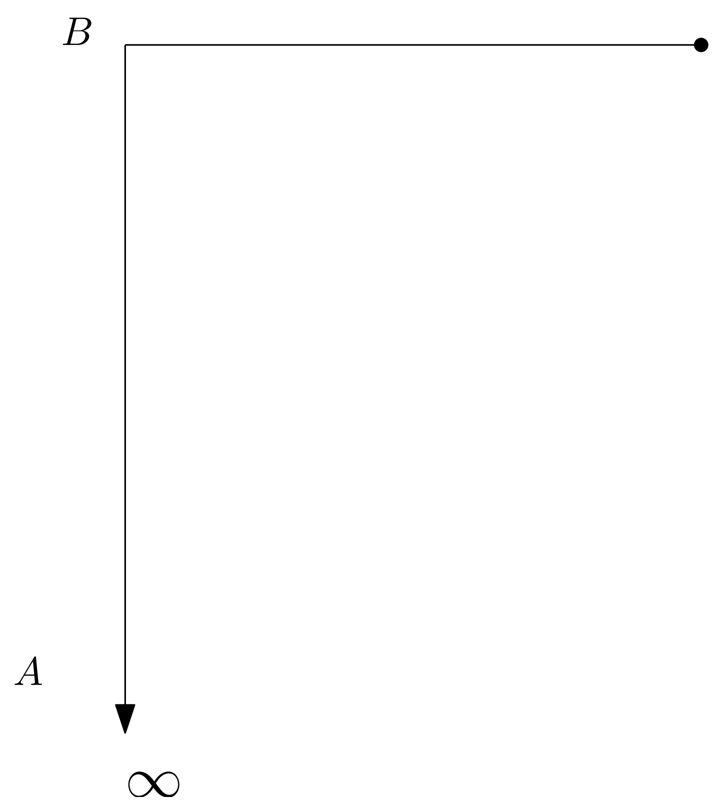
$$E = \frac{\rho_L}{4\pi\epsilon_0\rho}\left[\sin\alpha_1 a_\rho + (1-\cos\alpha_1)a_z\right] \tag{3.39}$$



3.2.5 CASE IV: SEMI- INFINITE LINE:

Lower end goes to infinity-field point coinciding with the upper end:
$\alpha_1 = 90^0,\ \alpha_2 = 0$

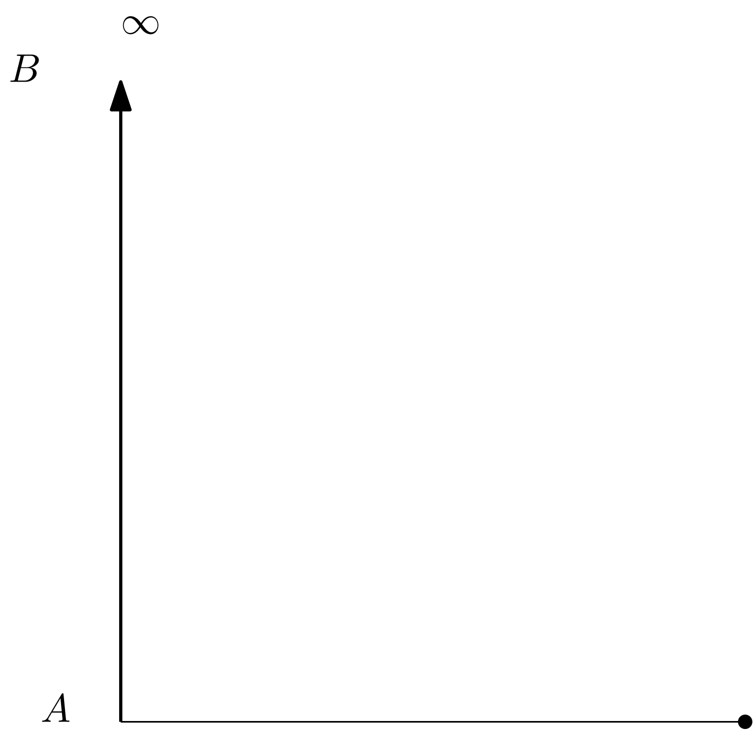
$$E = \frac{\rho_L}{4\pi\epsilon_0\rho}\left[a_\rho + a_z\right] \tag{3.40}$$



3.2.6 CASE V:SEMI- INFINITE LINE:

Upper end goes to infinity. Field point coincides with the lower end.
$\alpha_1 = 0,\ \alpha_2 = -90^0$

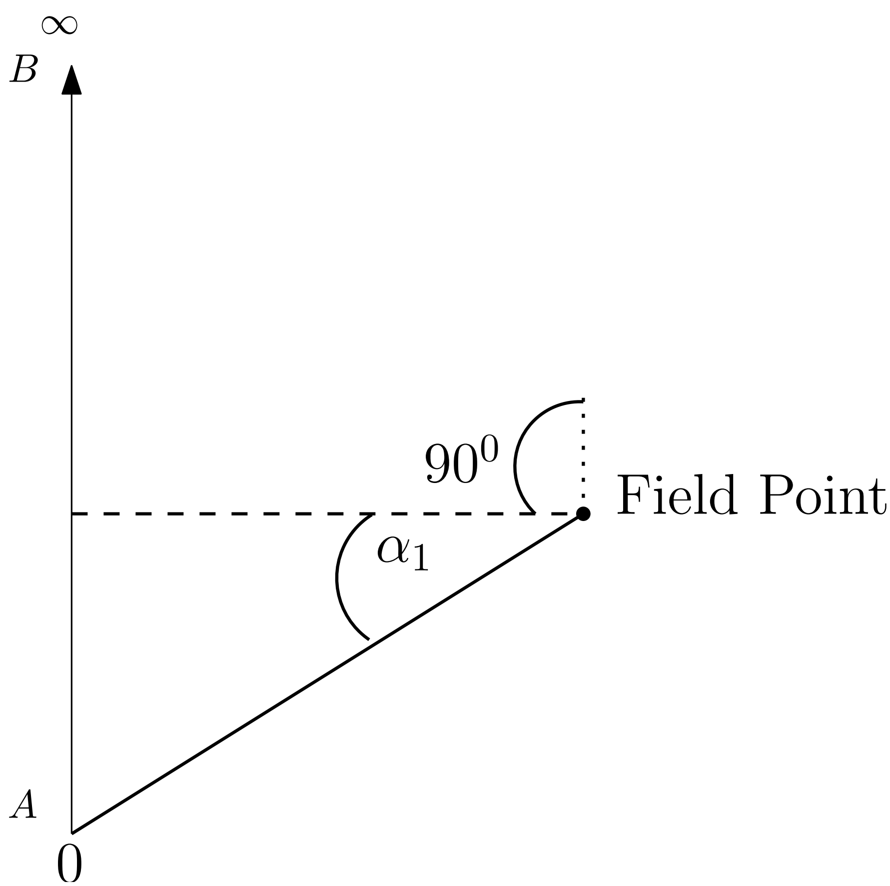
$$E = \frac{\rho_L}{4\pi\epsilon_0\rho}\left[a_\rho - a_z\right] \tag{3.41}$$



3.2.7 CASE VI: SEMI INFINITE LINE:

General point. Line extending from 0 to ∞ .
$\alpha_1 = \alpha_1,\ \alpha_2 = -90^0$

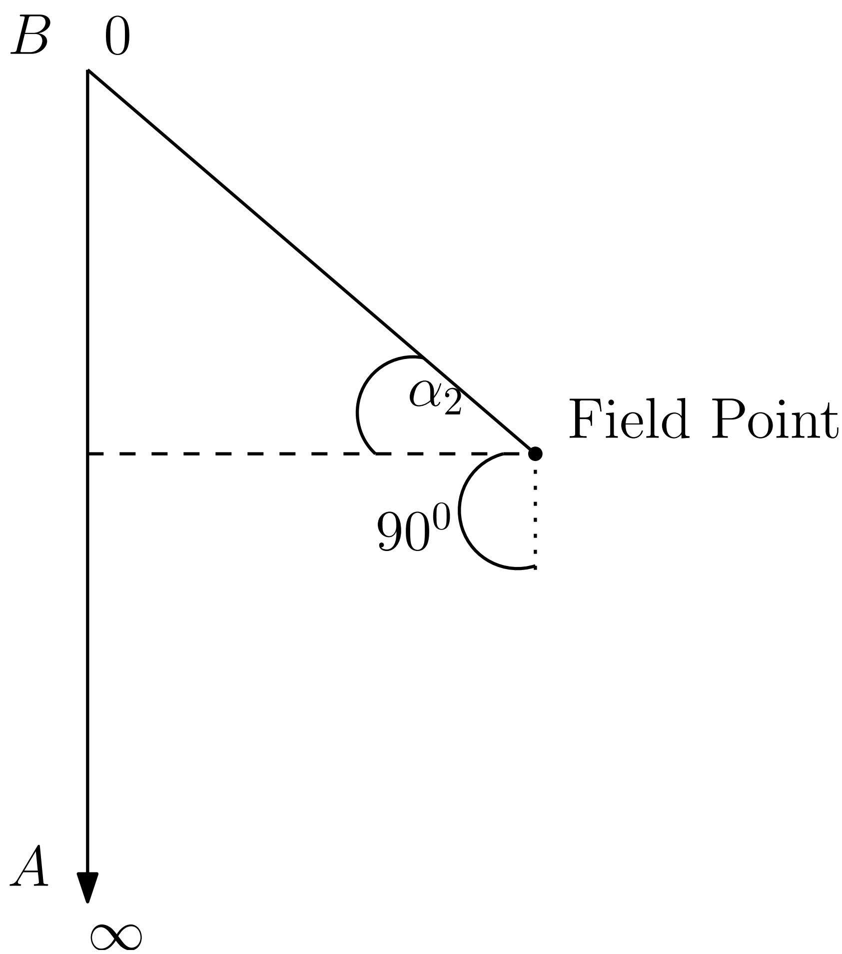
$$E = \frac{\rho_L}{4\pi\epsilon_0\rho}\left[(\sin\alpha_1 - 1)a_\rho - \cos\alpha_1 a_z\right] \tag{3.42}$$



3.2.8 CASE VII:SEMI-INFINITE LINE:

General point. Line extending from 0 , $-\infty$
$\alpha_1 = 90^0, \ \alpha_2 = \alpha_2$

$$E = \frac{\rho_L}{4\pi\epsilon_0\rho}\left[(1+\sin\alpha_2)a_\rho + \cos\alpha_2 a_z\right] \tag{3.43}$$



3.2.9 CIRCULAR RING OF CHARGE:

To find the field at any point on the axis of a circular ring of charge, whose axis coincides with the z axis. The charge density is $\rho_L C/m$. The radius of the ring is a m Consider a small differential length $dl = ad\phi$ on the ring. Consider a point $(0, 0, h)$. Th distance between the charge element and the point on the z axis is

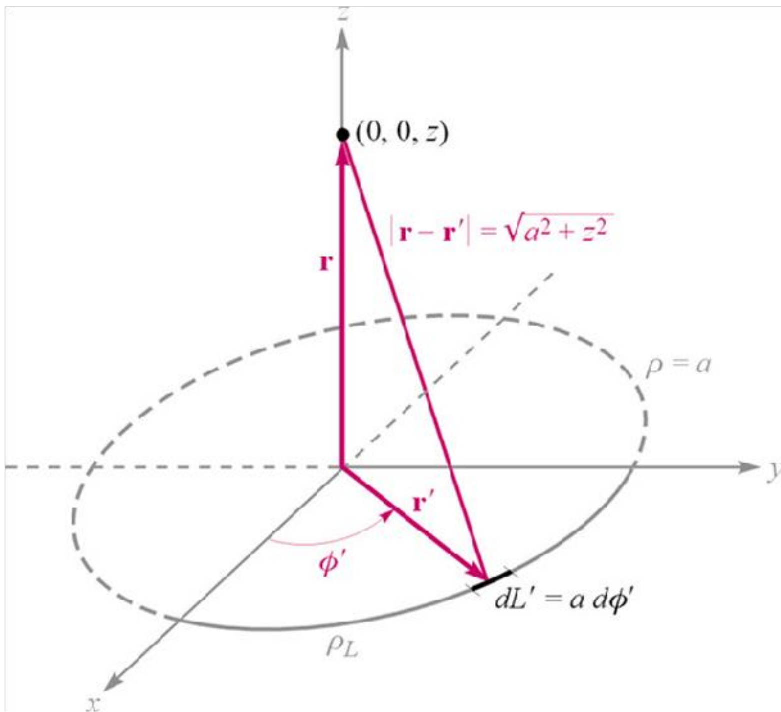


Figure 3.4: Ring of charge

$$R = -aa_\rho + ha_z \tag{3.44}$$

The total charge in the differential length element is $dQ = \rho_L a d\phi$

$$|R| = (a^2 + h^2)^{\frac{1}{2}} \tag{3.45}$$

$$a_R = \frac{R}{|R|^3} = \frac{-aa_\rho + ha_z}{(a^2 + h^2)^{\frac{3}{2}}} \tag{3.46}$$

$$E = \frac{\rho_L}{4\pi\epsilon_0}\int_{\phi=0}^{2\pi}\frac{(-aa_\rho + ha_z)}{(a^2 + h^2)^{\frac{3}{2}}}ad\phi \tag{3.47}$$

The above is a sum of two integrals and the first one over $0\,to\,2\pi$ is equal to zero as $a_\rho = \cos\phi a_x - \sin\phi a_y$ So the resulting integral is

$$E = \frac{\rho_L}{4\pi\epsilon_0}\frac{ah}{(a^2+h^2)^{\frac{3}{2}}}a_z\int_0^{2\pi} d\phi = \frac{\rho_L}{4\pi\epsilon_0}\frac{ah}{(a^2+h^2)^{\frac{3}{2}}}a_z \times 2\pi \quad (3.48)$$

The result is

$$E = \frac{\rho_L ah}{2\epsilon_0(a^2+h^2)^{\frac{3}{2}}}a_z \quad (3.49)$$

Maximum value of E:
To find the maximum value of E equate $\frac{dE}{dh}$ to zero. This gives $h = \pm\frac{a}{\sqrt{2}}$ As $a \to 0$ the ring behaves like a point charge. If the total charge on the ring is Q then $\rho_L = \frac{Q}{2\pi a}$ Then

$$E = \frac{Qh}{4\pi\epsilon_0[h^2+a^2]^{\frac{3}{2}}}a_z \quad (3.50)$$

as $a \to 0$

$$E = \frac{Q}{4\pi\epsilon_0 h^2}a_z \quad (3.51)$$

Same as the field of a point charge.

3.2.10 SURFACE CHARGE DISTRIBUTION:

Assume a disk of radius $'a'\,m$ with a surface charge density $\rho_s C/m^2$.Assume that the axis coincides with the $z-axis$. It Is required to find the field at any point $(0, 0, h)$ on the axis of the disk. Consider a small surface area element

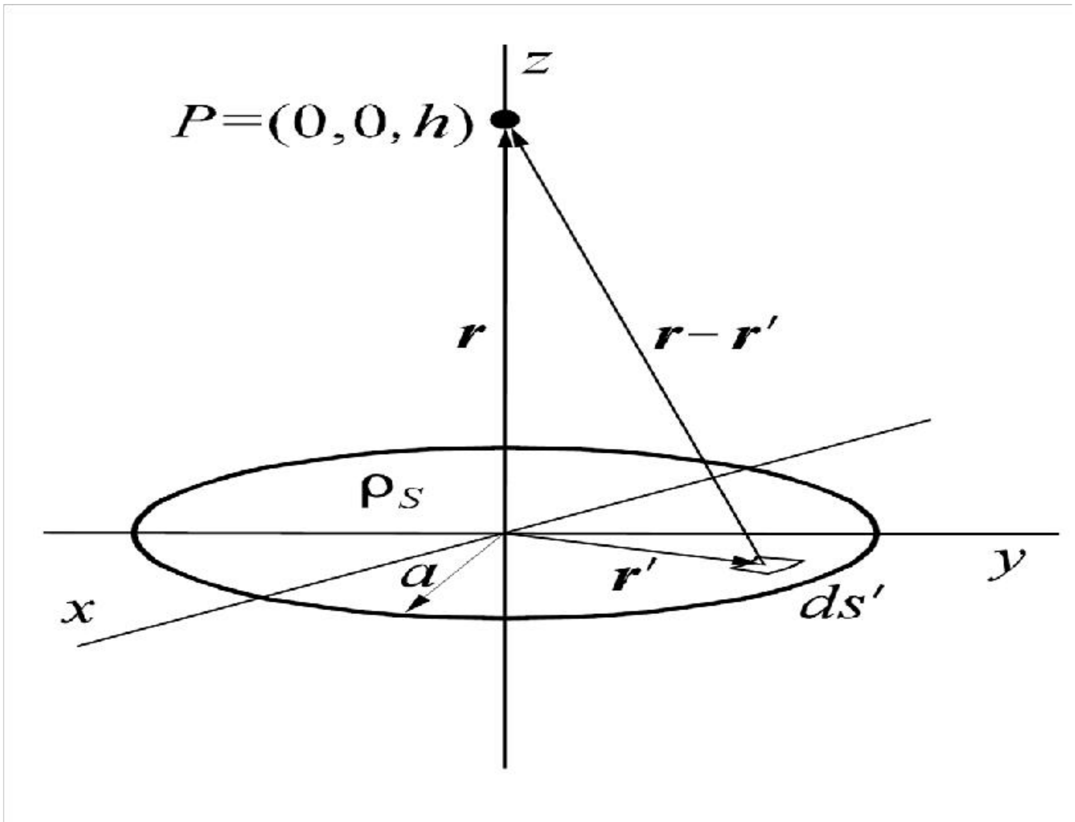


Figure 3.5: Disk of charge

$$ds = \rho d\rho d\phi \tag{3.52}$$

. The charge in that small differential area element is given by

$$dQ = \rho_s ds = \rho_s \rho d\rho d\phi \tag{3.53}$$

.The field because of this charge is given by

$$de = \frac{dQ}{4\pi\epsilon_0 R^2} a_R \tag{3.54}$$

where

$$R = -\rho a_\rho + h a_z \tag{3.55}$$

and

$$a_R = \frac{R}{|R|} = \frac{-\rho a_\rho + h a_z}{\sqrt{\rho^2 + h^2}} \tag{3.56}$$

Th e field because of the complete disk is

$$e = \int dE = \int_0^a \int_0^{2\pi} \left[\frac{\rho_s \rho d\rho d\phi}{4\pi\epsilon_0} \frac{\{-\rho a_\rho + h a_z)}{[\rho^2 + h^2]^{\frac{3}{2}}} \right] \tag{3.57}$$

The first term containing the unit vector a_ρ is zero integrated over $0\ to 2\pi$. The second term that remains is

$$E = \frac{\rho_s}{4\pi\epsilon_0} \int_0^a \int_0^{2\pi} \frac{h\rho d\rho d\phi}{[\rho^2 + h^2]^{\frac{3}{2}}} a_z \tag{3.58}$$

Integration by substitution of variable

$$\rho = h \tan\theta \tag{3.59}$$

results in

$$E = \frac{\rho_s}{2\epsilon_0} \left\{ 1 - \frac{h}{[h^2 + a^2]^{\frac{1}{2}}} \right\} a_z \tag{3.60}$$

As $a \to \infty$ the charge configuration tends to an infinite sheet of charge and the field is equal to

$$E = \frac{\rho_s}{2\epsilon_0} a_z \tag{3.61}$$

that is, E has only $z-$ component if the charge is in the $x - y$ plane. In general, for an infinite sheet of charge

$$\boxed{E = \frac{\rho_s}{2\epsilon_0} a_n}$$

where a_n is a unit vector normal to the sheet. From the above equation it can be noticed that the electric field is normal to the sheet and is independent of the distance between the sheet and the point of observation P .

In a parallel plate capacitor, the electric field existing between two plates having equal and opposite charges is given by

$$E = \frac{\rho_s}{2\epsilon_0}a_n + \frac{-\rho_s}{2\epsilon_0}(-a_n) = \frac{\rho_s}{\epsilon_0}a_n \tag{3.62}$$

3.3 ENERGY AND POTENTIAL

3.3.1 ENERGY EXPENDED IN MOVING A POINT CHARGE IN AN ELECTRIC FIELD:

Suppose we wish to move a charge Q a distance dL in an electric field E. The force on Q due to the electric field is

$$\boxed{F = QE}$$

where the subscript indicates that the force is due to the electric field. The component of this force in the direction of dL which an external force has to overcome is

$$F_{EL} = F \bullet a_L = QE \bullet a_L \tag{3.63}$$

where a_L is a unit vector in the direction of dL The external force that must be applied is equal and opposite to the above force

$$F_{appl} = -QE \bullet a_L \tag{3.64}$$

or

$$dW = -QE \bullet dL \tag{3.65}$$

The work done in moving a charge Q a finite distance is determined by integrating

$$W = -Q\int_{initial}^{final} E \bullet dL \tag{3.66}$$

where the path must be specified before the integration is performed.Th e charge is assumed to be at rest at both its initial and final positions.

Example: Given the electric field $E = \frac{1}{z^2}\left(8xyza_x + 4x^2za_y - 4x^2ya_z\right) V/m.$ Find the differential amount of work done in moving a $6 - nC$ charge a distance $2\mu m$, starting at $P(2, -2, 3)$ and proceeding in the direction $a_L =: i) -\frac{6}{7}a_x + \frac{3}{7}a_y + \frac{2}{7}a_z; ii)\frac{6}{7}a_x - \frac{3}{7}a_y - \frac{2}{7}a_z; iii)\frac{3}{7}a_x + \frac{6}{7}a_y.$

3.3.2 The LINE INTEGRAL:

The integral expression for the work done in moving a point charge Q from one position to another, equation: is an example of a line integral. The procedure for evaluating the integral is shown in fig: , where a path has been chosen from an initial point B to a final point A and uniform electric field is selected for simplicity.

The path is divided into six segments , $\triangle L_1, \triangle L_2 \ldots \triangle L_6$

, and the components of E along each segment is denoted by $E_{L1}, E_{L2} \ldots E_{L6}$

The involved in moving a charge Q from B to A is

$$w = -Q\left[E_{L1}\triangle L_1 + E_{L2}\triangle L_2 + \ldots E_{L6}\triangle L_6\right] \tag{3.67}$$

or using vector notation

$$w = -Q\left[E_{L1} \bullet \triangle L_1 + E_{L2} \bullet \triangle L_2 + \ldots E_{L6} \bullet \triangle L_6\right] \tag{3.68}$$

as the field is uniform $E_1 = E_2 = \ldots E_6$

$$w = -QE \bullet [\triangle L_1 + \triangle L_2 + \ldots + \triangle L_6] \tag{3.69}$$

The sum of all these small vector length segments is equal to the vector directed from the initial point B to the final point A, and is denoted by L_{BA} . Therefore

$$W = -QE \bullet L_{BA} \tag{3.70}$$

This result for a uniform field can be written as an integral

$$W = -Q \int_B^A E \bullet dl \tag{3.71}$$

Now we can define potential difference V as the work done (by a n external agency) in moving a unit positive charge from one point to another in an electric field

$$V = \frac{W}{Q} = -\int_{initial}^{final} E \bullet dl \tag{3.72}$$

V_{AB} signifies the potential difference between points A and B and is the work done in moving the unit positive charge from B (last named) to A (the first named). In determining V_{AB} , B is the initial point and A is the final point. Potential difference is measured in Joules /Coulomb, which is commonly called as Volt. hence

$$\boxed{V_{AB} = -\int_B^A E \bullet dl \quad V} \tag{3.73}$$

and V_{AB} is positive if work is done in carrying a positive charge from B to A .

3.3.2.1 THE POTENTIAL FIELD OF A POINT CHARGE:

To find the potential difference between points A and B at radial distances r_A and r_B from a point charge Q, assume that the origin is at Q then

$$E = E_r a_r = \frac{Q}{4\pi\epsilon_0 r^2} a_r \tag{3.74}$$

and

$$dL = dr a_r \tag{3.75}$$

we have

$$V_{AB} = -\int_B^A E \bullet dL = -\int_{r_A}^{r_B} \frac{Q}{4\pi\epsilon_0 r^2} dr = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right) = V_A - V_B \tag{3.76}$$

If $r_A > r_B$, the potential difference V_{AB} is positive indicating that energy is expended by external agent in bringing the positive charge from r_B to r_A . See figure below

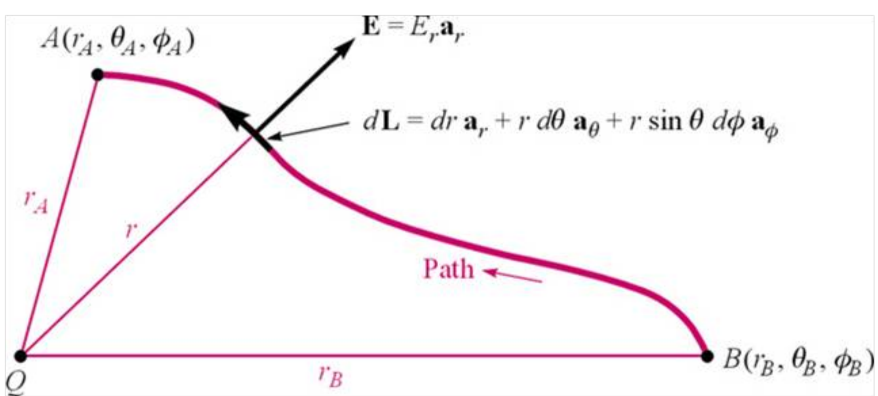


Figure 3.6: Potential in the field of a point charge

To speak of the potential or absolute potential, of a point, rather than the potential difference, an agreement must be reached to measure the potential with respect to a specified reference point which should be considered to have zero potential. The most universal zero reference point is "ground". Theoretically it is represented by an infinite plane at zero potential. Another widely used reference point is infinity. This usually appears in theoretical problems. Also it is necessarily agreed that V_A and $V_B$shall have the same zero reference point.

From the above figure it can also be seen that the potential difference does not depend on path of integration, but depends only on the distance of each point from the charge(that is, only on the end points).

3.3.2.2 POTENTIAL FIELD BECAUSE OF A GROUP OF CHARGES:

The potential at a point has been defined as the work done in bringing a unit positive charge from the zero reference to the point. Thus the potential of a single point charge Q_1 located at r_1 at a point distant r from the origin with zero reference at infinity

$$V(r) = \frac{Q_1}{4\pi\epsilon_0 \left| r - r_1 \right|} \tag{3.77}$$

The potential due to n charges is at this point is given by

$$V(r) = \frac{Q_1}{4\pi\epsilon_0 \left| r - r_1 \right|} + \frac{Q_2}{4\pi\epsilon_0 \left| r - r_2 \right|} + \ldots\ldots + \frac{Q_n}{4\pi\epsilon_0 \left| r - r_n \right|} \tag{3.78}$$

$$V(r) = \sum_{m=1}^{n} \frac{Q_m}{4\pi\epsilon_0 |r - r_m|} \tag{3.79}$$

If each point charge is now represented as a small element of a continuous volume charge distribution $\rho_v \triangle v$, then

$$V(r) = \int_{vol} \frac{\rho_v(r^{'})dv'}{4\pi\epsilon_0 |r - r'|} \tag{3.80}$$

For line charge and also for surface charge distribution the respective expressions are

$$V(r) = \int \frac{\rho_L(r^{'})dL'}{4\pi\epsilon_0 |r - r'|}$$

$$V(r) = \int_s \frac{\rho_v(r^{'})ds'}{4\pi\epsilon_0 |r - r'|}$$

3.3.2.3 THE POTENTIAL FIELD OF A RING OF UNIFORM LINE CHARGE DENSITY:

To find V on the axis of a uniform line charge ρ_L in the form a ring of radius a , in the $z = 0$ plane as shown in figure.

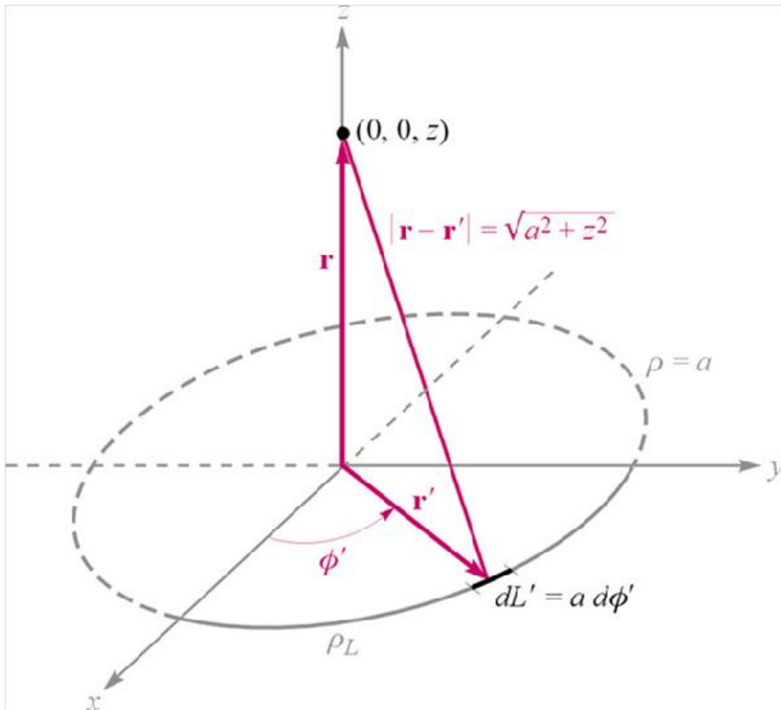


Figure 3.7: Ring of charge

we have

$$dL' = ad\phi',\ r = za_z,\ r' = aa_\rho,\ \left|r - r'\right| = \sqrt{a^2 + z^2} \qquad (3.81)$$

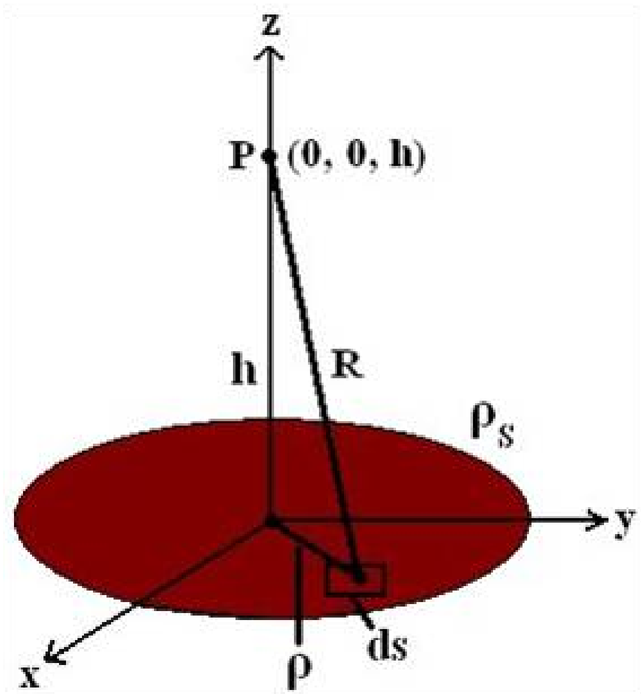
and

$$V = \int_0^{2\pi} \frac{\rho_L a d\phi'}{4\pi\epsilon_0\sqrt{a^2+z^2}} = \frac{\rho_L a}{2\epsilon_0\sqrt{a^2+z^2}} \qquad (3.82)$$

so potential at any point on the axis of a uniformly charged ring is

$$\boxed{V = \frac{\rho_L a}{2\epsilon_0\sqrt{a^2+z^2}}}$$

3.3.2.4 POTENTIAL AT ANY POINT ON THE AXIS OF UNIFORMLY CHARGED DISC:



$$
\begin{aligned}
dq &= \rho_s 2\pi r dr \\
dV &= \frac{dq}{4\pi\epsilon_0 R} \\
R &= \sqrt{r^2 + h^2} \\
dV &= \frac{\rho_s 2\pi r dr}{4\pi\epsilon_0\sqrt{r^2 + h^2}} \\
V &= \frac{\rho_s}{2\epsilon_0}\int_0^a \frac{rdr}{\sqrt{r^2 + h^2}} \\
V &= \frac{\rho_s}{2\epsilon_0} h\left(\sec\alpha - 1\right) + C
\end{aligned}
$$

If $C = 0$ at $z = \infty$

$$V = \frac{\rho_s}{2\epsilon_0} h\left(\sec\alpha - 1\right) \tag{3.83}$$

Example:

A charge of $Q\,C$ is distributed homogeneously over the surface of a sphere of radius R meters. The sphere is in vacuum . Find the potential Vas a function of distance r from the center of the sphere for $0 \leq r \leq \infty$. $V(\infty) = 0$.

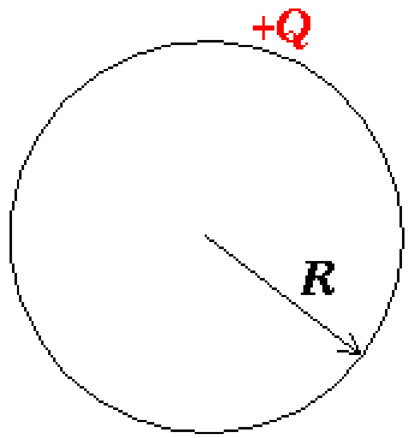
Answer:

Outside the sphere

$$\begin{aligned} V &= -\int_{\infty}^{r} E \bullet dr \quad r > R \\ V &= -\frac{Q}{4\pi\epsilon_0}\int_{\infty}^{r} \frac{dr}{r^2} = \frac{Q}{4\pi\epsilon_0 r} \end{aligned}$$

Inside the sphere:

$$V(r) = -\int_{\infty}^{r} E \bullet dr = -\frac{1}{4\pi\epsilon_0}\left(\int_{\infty}^{r} \frac{Q}{r^2}dr - \int_{r}^{R} (0)dr\right) = \frac{1}{4\pi\epsilon_0}\frac{Q}{R} \quad (r < R) \tag{3.84}$$



3.3.2.5 POTENTIAL AND ELECTRIC FIELD OF A DIPOLE:

An electric dipole is formed by two point charges of equal magnitude and opposite sign $(+Q, -Q)$ separated by a short distance d. The potential at the point P due to the electric dipole is found using superposition.

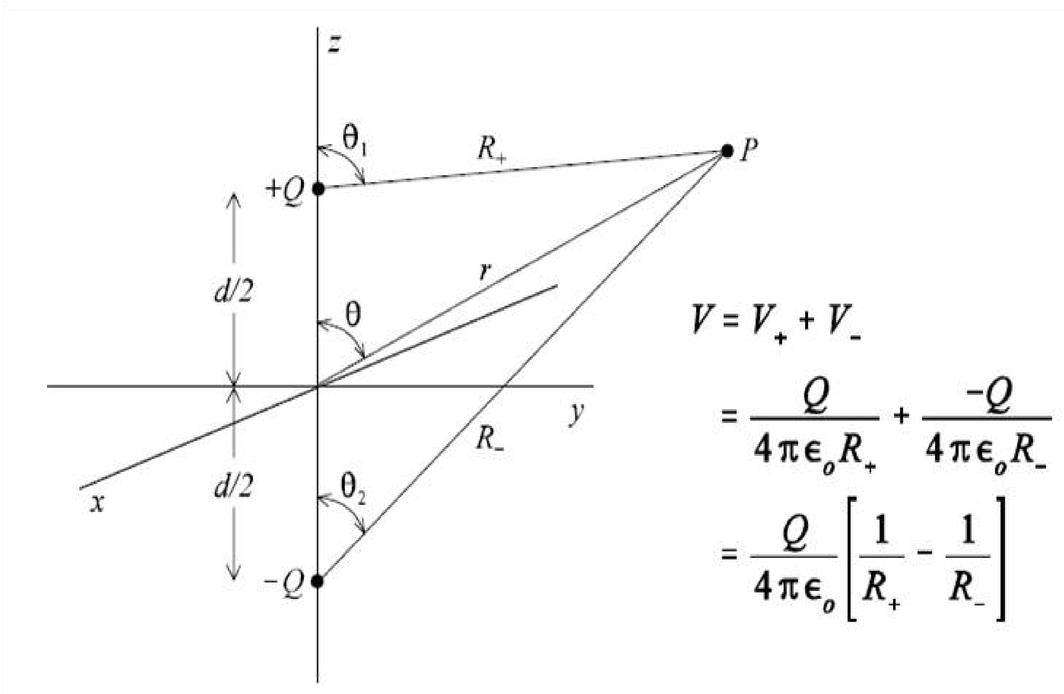


Figure 3.8: Dipole

If the field point P is moved a large distance from the electric dipole (in what is called the far field, $r \gg d$ the lines connecting the two charges and the coordinate origin with the field point become nearly parallel.

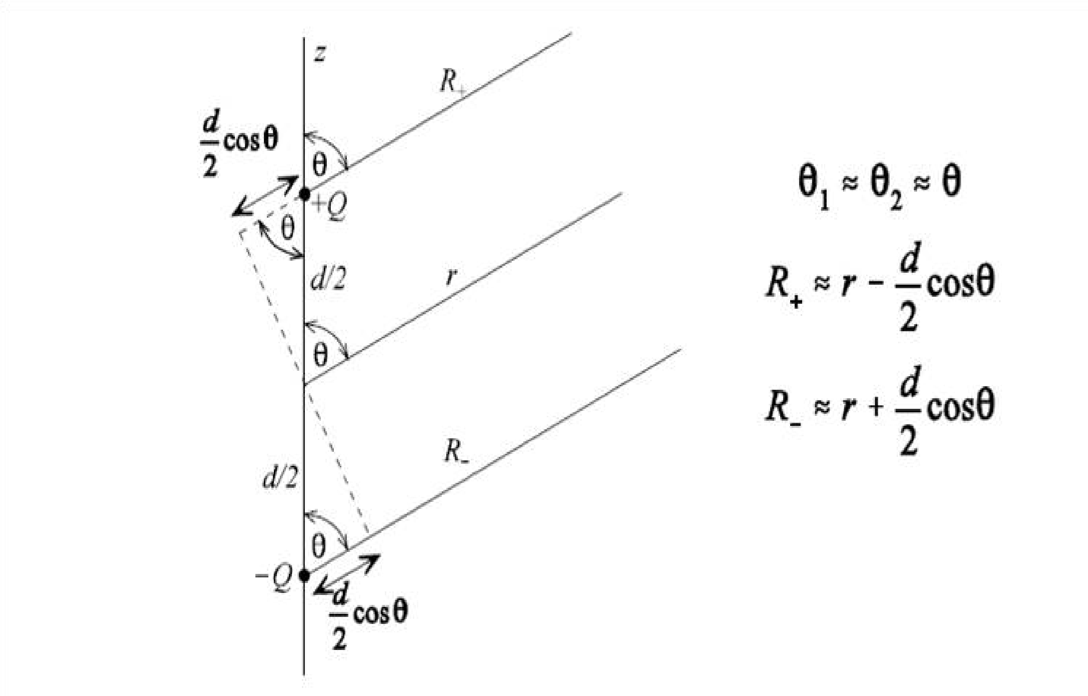


Figure 3.9: Far field approximation

$$V \approx \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r-\frac{d}{2}\cos\theta} - \frac{1}{r+\frac{d}{2}\cos\theta}\right]$$

$$V \approx \frac{Q}{4\pi\epsilon_0}\left[\frac{\left(r+\frac{d}{2}\cos\theta\right)-\left(r-\frac{d}{2}\cos\theta\right)}{\left(r^2-\frac{d^2}{4}\cos^2\theta\right)}\right]$$

as $r \gg d$ the far field is

$$V = \frac{Qd\cos\theta}{4\pi\epsilon_0 r^2} \tag{3.85}$$

The electric field produced by the electric dipole is found by taking the gradient of the potential.

$$
\begin{aligned}
V = -\nabla E &= -\left[\frac{\partial V}{\partial r}a_r + \frac{1}{r}\frac{\partial V}{\partial \theta}a_\theta\right] \\
&= -\frac{Qd}{4\pi\epsilon_0}\left[\cos\theta\frac{\partial}{\partial r}\left(\frac{1}{r^2}\right)a_r + \frac{1}{r^3}\frac{\partial}{\partial\theta}(\cos\theta)a_\theta\right] \\
&= -\frac{Qd}{4\pi\epsilon_0}\left[\cos\theta\left(-\frac{2}{r^3}\right)a_r + \frac{1}{r^3}(-\sin\theta)a_\theta\right] \\
&= \frac{Qd}{4\pi\epsilon_0 r^3}\left[2\cos\theta a_r + (\sin\theta)a_\theta\right]
\end{aligned}
$$

If the vector dipole moment is defined as

$$P = pa_p = Qda_p \tag{3.86}$$

where a_p points from $+Q$ to $-Q$. The dipole potential and electric field may be written as

$$
\boxed{
\begin{aligned}
V &= \frac{Qd\cos\theta}{4\pi\epsilon_0 r^2} = \frac{P \bullet a_r}{4\pi\epsilon_0 r^2} \\
E &= \frac{P}{4\pi\epsilon_0 r^3}\left[2\cos\theta a_r + \sin\theta a_\theta\right]
\end{aligned}
}
$$

Note that the potential and electric field of the electric dipole decay faster than those of a point charge. For an arbitrarily located, arbitrarily oriented dipole, the potential can be written as

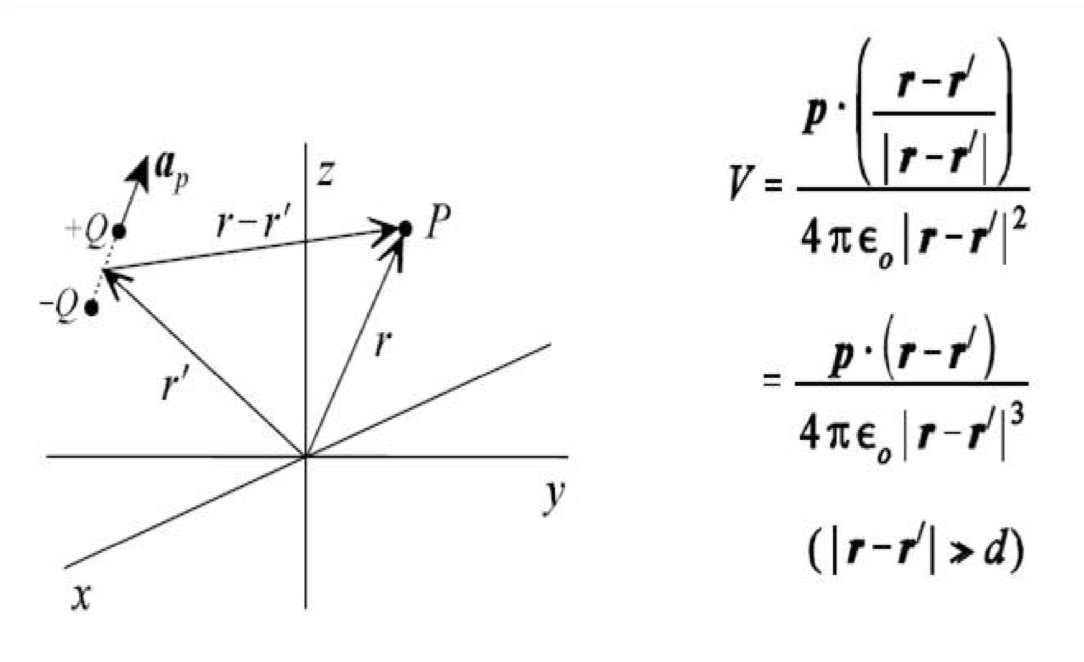


Figure 3.10: Arbitrarily placed dipole

3.3.3 ENERGY STORED IN AN ELECTROSTATIC FIELD:

The amount of work necessary to assemble a group of point charges equals the total energy (W_e) stored in the resulting electric field.

Example (3 point charges): Given a system of 3 point charges, we can determine the total energy stored in the electric field of these point charges by determining the work performed to assemble the charge distribution. We first define V_{mn} as the absolute potential at P_m due to point charge Q_n.

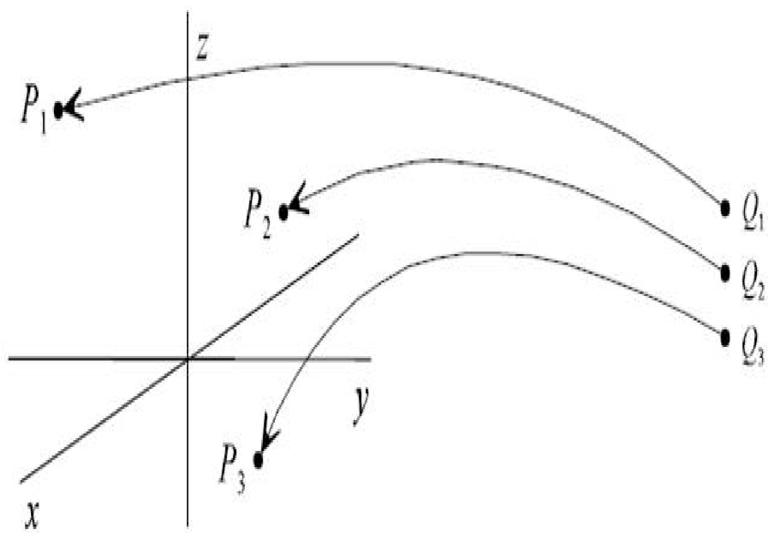


Figure 3.11: Energy to move point charges

1. Bring Q_1 to P_1(no energy required).
2. Bring Q_2 to P_2 (work $= Q_2V_{21}$).
3. Bring Q_3 to P_3 (work $= Q_3V_{31} + Q_3V_{32}$)

The total work done $W_e = 0 + Q_2V_{21} + Q_3V_{31} + Q_3V_{32}$

If we reverse the order in which the charges are assembled, the total energy required is the same as before.

1. Bring Q_3 to P_3 (No energy required)
2. Bring Q_2 to P_2 (work$=Q_2V_{23}$)
3. Bring $Q_1 to P_1$ (work done $= Q_1V_{12} + Q_1V_{13}$)

Total work done $W_e = 0 + Q_2V_{23} + Q_1V_{12} + Q_1V_{13}$

Adding the above two equations

$$2W_e = Q_1V_{12} + Q_1V_{13} + Q_2V_{21} + Q_2V_{23} + Q_3V_{31} + Q_3V_{32} \quad (3.87)$$

$$W_e = \frac{1}{2}\left[(Q_1(V_{12} + V_{13}) + Q_2(V_{21} + V_{23}) + Q_3(V_{31} + V_{32})\right] = \frac{1}{2}\left[Q_1V_1 + Q_2V_2 + Q_2V_3 \right. \quad (3.88)$$

where V_m is the total absolute potential at P_m affecting Q_m.

In general, for a system of N point charges, the total energy in the electric field is given by

$$W_e = \frac{1}{2}\sum_{k=1}^{N} Q_kV_k \quad (3.89)$$

For line, surface or volume charge distributions, the discrete sum total energy formula above becomes a continuous sum (integral) over the respective charge distribution. The point charge term is replaced by the appropriate differential element of charge for a line, surface or volume distribution: $\rho_L dL$, $\rho_s ds$ or $\rho_v dv$. The overall potential acting on the point charge Q_k due to the other point charges (V_k) is replaced by the overall potential (v) acting on the differential element of charge due to the rest of the charge distribution. The total energy expressions becomes

$$W_e = \frac{1}{2}\int_L \rho_L dL \qquad (Line\ Charge) \quad (3.90)$$

$$W_e = \frac{1}{2}\int_s \rho_s ds \qquad (Surface\ Charge) \quad (3.91)$$

$$W_e = \frac{1}{2}\int\limits_v \rho_v dv \qquad (Volume\ Charge) \tag{3.92}$$

If a volume charge distribution ρ_v of finite dimension is enclosed by a spherical surface S_0 of radius r_0, the total energy associated with the charge distribution is given by

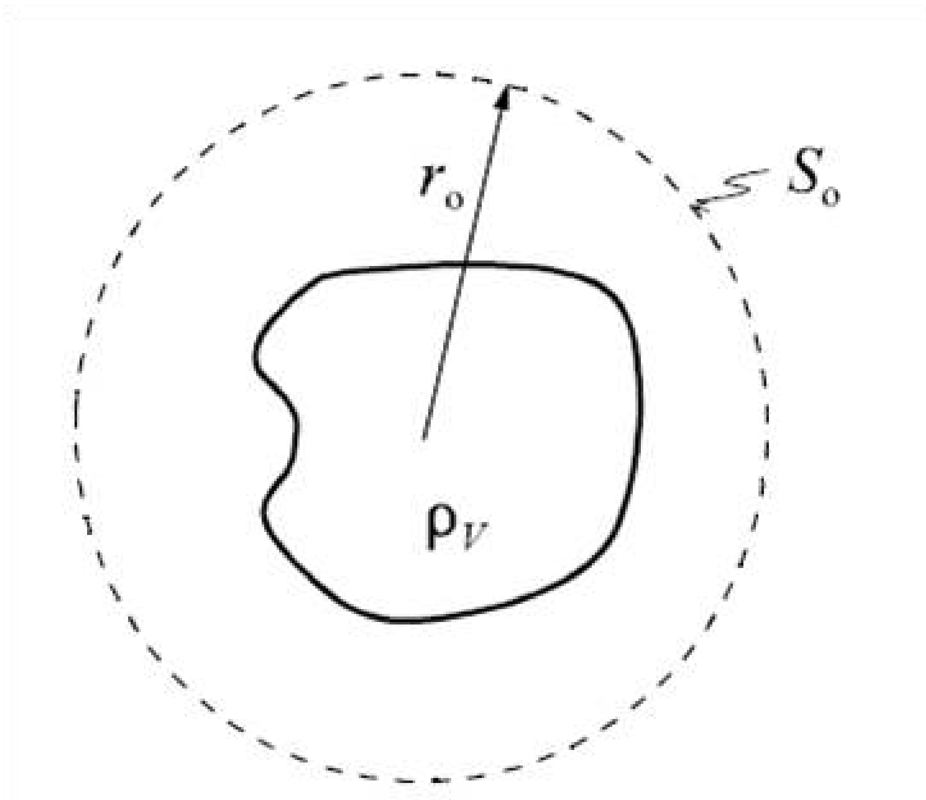


Figure 3.12: Distribution of volume charge

$$W_e = \lim_{r_0 \to \infty} \left[\frac{1}{2} \int_v \rho_v V dv \right] = \lim_{r_0 \to \infty} \left[\frac{1}{2} \int (\nabla \bullet D) V dv \right] \quad (3.93)$$

Using the following vector identity,

$$(\nabla \bullet D)V = \nabla \bullet (VD) - D \bullet \nabla V \quad (3.94)$$

the expression for the total energy can be written as

$$W_e = \lim_{r_0 \to \infty} \left[\frac{1}{2} \int_v [\nabla \bullet (VD)] dv \right] - \lim_{r_0 \to \infty} \left[\frac{1}{2} \int_v (D \bullet \nabla V) dv \right] \quad (3.95)$$

If we apply the divergence theorem to the first integral, we find

$$W_e = \lim_{r_0 \to \infty} \left[\frac{1}{2} \oiint VD \bullet ds \right] - \lim_{r_0 \to \infty} \left[\frac{1}{2} \int_v (D \bullet \nabla V) dv \right] \quad (3.96)$$

For each equivalent point charge ($\rho_v dv$) that makes up the volume charge distribution, the potential contribution on S_0 varies as r^{-1} and electric flux density (and electric field) contribution varies as r^{-2}. Thus, the product of the potential and electric flux density on the surface So varies as r^{-3}. Since the integration over the surface provides a multiplication factor of only r^2, the surface integral in the energy equation goes to zero on the surface S_0 of infinite radius. This yields where the integration is applied over all space. The divergence term in the integrand can be written in terms of the electric field as

$$E = -\nabla V \quad (3.97)$$

such that the total energy (J) in the electric field is

$$W_e = \frac{1}{2}\iiint_v D \bullet E dv = \frac{1}{2}\iiint_v \epsilon_0 (E \bullet E) dv = \frac{1}{2}\iiint_v \epsilon_0 E^2 dv \tag{3.98}$$

This can also be expressed as

$$\frac{dW_e}{dv} = \frac{1}{2}\epsilon_0 E^2 \tag{3.99}$$

$\frac{dW_e}{dv}$ is called the energy density and is given in J/m^3.

3.4 GAUSS'S LAW:

Johann Carl Friedrich Gauss: (30 April 1777 – 23 February 1855) was a German mathematician and physical scientist who contributed significantly to many fields, including number theory, statistics, analysis, differential geometry, geodesy, geophysics, electrostatics, astronomy and optics. Sometimes referred to as the Princeps mathematicorum (Latin, "the Prince of Mathematicians" or "the foremost of mathematicians") and "greatest mathematician since antiquity", Gauss had a remarkable influence in many fields of mathematics and science and is ranked as one of history's most influential mathematicians.He referred to mathematics as "the queen of sciences".

3.4.1 ELECTRIC FLUX DENSITY:

Consider a set of concentric metallic spheres, the outer one consisting of two hemi-spheres which could be firmly clamped together.

1. with the equipment dismantled, the inner sphere was given a known positive charge.
2. the hemispheres were then clamped together around the charged sphere with about 2cm of dielectric material between them
3. The outer sphere was discharged by connecting it momentarily to ground

4. The outer sphere was separated carefully, using tools made up of insulating material in order not to disturb the induced charge on each hemisphere, and the negative induced charge on each hemisphere was measured. See fig. below

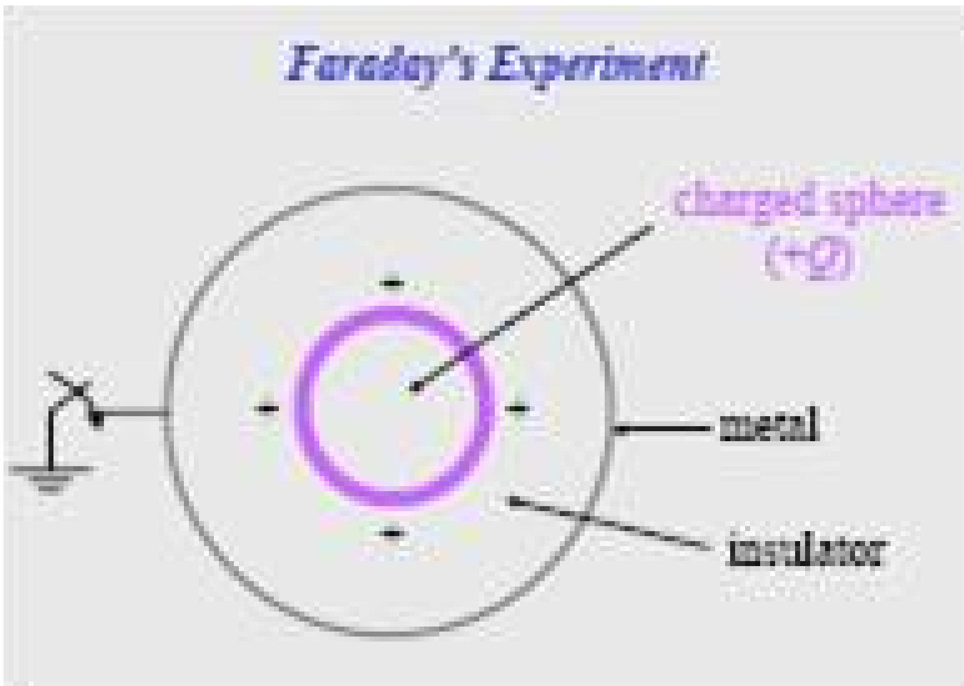


Figure 3.13: Faraday's experiment

It can be found that the total charge on the outer charge was equal in magnitude to the original charge placed on the inner sphere and this was true regardless of the dielectric material separating the two spheres. There was some sort of a "displacement" from the inner sphere to the outer sphere which was independent of the medium, this as the displacement flux density or electrical flux. Electrical flux is represented by ψ

$$\boxed{\psi = Q}$$

and the electrical flux is measured in Coulombs. Electric flux density is represented by the letter D because of the name that was given initially "Displacement flux density". The electrical flux density D is a vector and is a member of the flux density class of vector fields. The direction of D at a point is the direction of the flux lines at that point, and the magnitude is given by the number of flux lines crossing a surface normal to the lines dived by the surface area. Refer to the fig. below

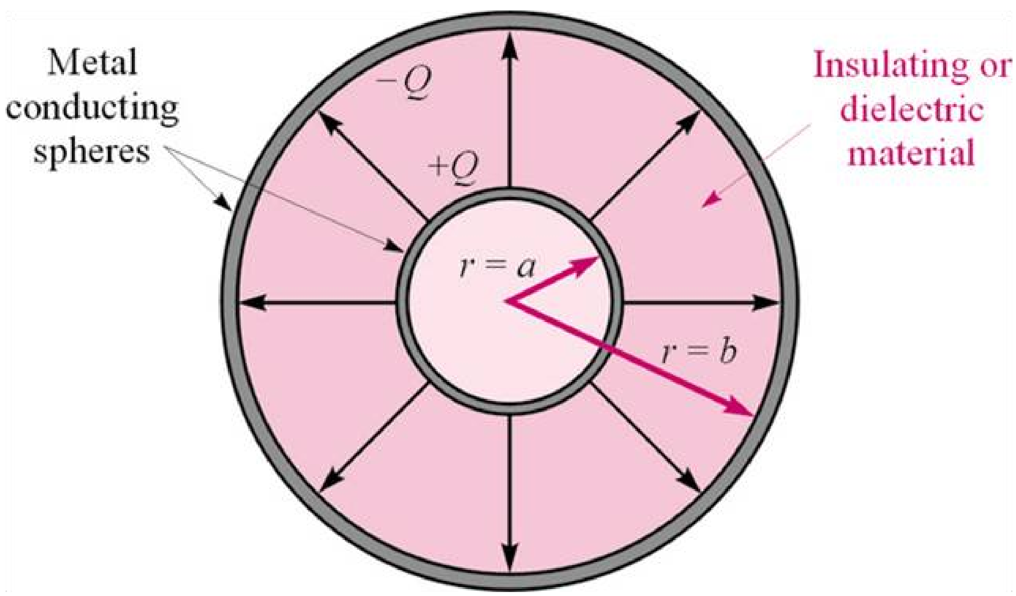


Figure 3.14: The electric flux in the region between a pair of charged concentric spheres

.

The electric flux density is in the radial direction and has a value of

$$D|_{r=a} = \frac{Q}{4\pi a^2} a_r$$
$$D|_{r=b} = \frac{Q}{4\pi b^2} a_r$$

and at a radial distance of r, $a \leq r \leq b$

$$D = \frac{Q}{4\pi r^2} a_r \tag{3.100}$$

If we compare this result with the equation for electric field intensity of a point charge in free space

$$E = \frac{Q}{4\pi \epsilon_0 r^2} a_r \tag{3.101}$$

So in free space

$$\boxed{D = \epsilon_0 E}$$

For a general charge distribution E and D are given by

$$E = \int_v \frac{\rho_v dv}{4\pi \epsilon_0 R^2} a_R$$
$$D = \int_v \frac{\rho_v dv}{4\pi R^2} a_R$$

The fig. below shows clearly what is the flux passing through an open surface.

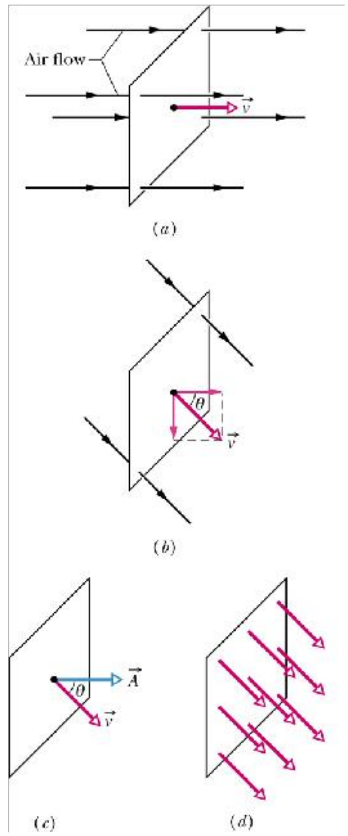


Figure 3.15: Electric flux through an open surface

3.4.2 GAUSS'S LAW:

From the above it can be seen that the electric flux passing through any imaginary spherical surface lying between the two conducting spheres is equal to the charge enclosed within that imaginary surface. This enclosed charge can be a charge that is distributed on the surface of the inner sphere, or it may be concentrated as a point charge at the center of the imaginary sphere. also as one Coulomb of charge produces one coulomb of flux, the inner conductor might as well be a cube or a queer shaped metal piece and still the total induced charge on the outer sphere would still be the same. The flux distribution will no longer be the same as the previous symmetrical distribution, but it will be some unknown distribution. If the outer hemisphere is replaced by a closed surface of any odd shape, still the result will be the same. The generalization of this concept leads to the following statement which is known as **Gauss law:**

The electrical flux ψ passing through any closed surface is equal to the total charge enclosed by that surface.

Gauss's law constitutes one of the fundamental laws of electromagnetism.

3.4.2.1 GAUSS'S LAW AND MAXWELL'S EQUATION:

If the flux emanating from a closed surface is ψ then

$$\psi = Q_{enc} \tag{3.102}$$

that is

$$\psi = \oint_s d\psi = \oint_s D \bullet ds = Q = \int_v \rho_v dv$$

$$\oint_s D \bullet ds = \int_v \rho_v dv \tag{3.103}$$

By applying divergence theorem to the first term in the above equation

$$\oint_s D \bullet ds = \int_v (\nabla \bullet D) dv = \int_v \rho_v dv \tag{3.104}$$

which gives

$$\nabla \bullet D = \rho_v \tag{3.105}$$
$$\oint_s D \bullet ds = \int_v \rho_v dv \tag{3.106}$$

which is the first of the four Maxwell's equations both in differential and integral form.

3.4.2.2 POTENTIAL GRADIENT:

The electric field at any general point is given by

$$E = \frac{Q}{4\pi\epsilon_0} \frac{\overrightarrow{r} - \overrightarrow{r'}}{|r - r'|^3} \tag{3.107}$$

Every expression for E whether it is because of a point charge or because of a general charge distribution contains the term

$$\frac{\overrightarrow{r} - \overrightarrow{r'}}{|r - r'|^3} \tag{3.108}$$

hence we want to find the curl of the above quantity and show how E is related to V the potential. From vector calculus

$$\nabla \times \frac{\overrightarrow{r} - \overrightarrow{r'}}{|r - r'|^3} = \frac{1}{|r - r'|^3} \nabla \times (\overrightarrow{r} - \overrightarrow{r}') + \left[\nabla \frac{1}{|r - r'|^3}\right] \times [\overrightarrow{r} - \overrightarrow{r}']$$

$$\nabla \times (\overrightarrow{r} - \overrightarrow{r}') = 0$$

$$\nabla \frac{1}{\left|\overrightarrow{r} - \overrightarrow{r'}\right|} = -3 \frac{\overrightarrow{r} - \overrightarrow{r}'}{|r - r'|^5}$$

The above results together with the observation that the cross product of a vector with a parallel vector is zero, is sufficient to prove that

$$\nabla \times \frac{\overrightarrow{r} - \overrightarrow{r}'}{|r - r'|^3} = 0 \tag{3.109}$$

So this shows that the curl of the electric field is zero. Then from vector calculus we know that if the curl of a vector is zero then the vector can be expressed as the gradient of a scalar point function. Also it can be seen that

$$E = \frac{Q}{4\pi\epsilon_0} \frac{\overrightarrow{r} - \overrightarrow{r'}}{|r - r'|^3} = -\nabla' \left[\frac{Q}{4\pi\epsilon_0} \frac{1}{|r - r'|}\right] \tag{3.110}$$

where

$$V = \frac{Q}{4\pi\epsilon_0} \frac{1}{|r - r'|} \tag{3.111}$$

is the scalar potential. Hence E and V are related by

$$\boxed{E = -\nabla V}$$

3.4.2.3 Static Electric Field And The Curl:

It is seen that

$$\oint E \bullet dl = 0 \tag{3.112}$$

and

$$E = -\nabla V \tag{3.113}$$

So if we apply the Stoke's theorem

$$\oint E \bullet dl = \int_s (\nabla \times E) \bullet ds = 0 \tag{3.114}$$

This is true for any ds , so

$$\boxed{\nabla \times E = 0 \quad (3.115)}$$

So the second of the Maxwell's equations, both in integral and differential form, which describe the electric field is

$$\oint E \bullet dl = 0 \tag{3.116}$$

$$\nabla \times E = 0 \tag{3.117}$$

The Maxwell's equations which describe the static electric field are given below

Differential Form	Integral Form
$\nabla \bullet D = \rho_v$	$\oint_s D \bullet ds = \int_v \rho_v dv$
$\nabla \times E = 0$	$\oint_L E \bullet dl = 0$

Table 3.1: Maxwell's Equations

3.4.3 Applications

3.4.3.1 Electric Forces in Biology

Classical electrostatics has an important role to play in modern molecular biology. Large molecules such as proteins, nucleic acids, and so on—so important to life—are usually electrically charged. DNA itself is highly charged; it is the electrostatic force that not only holds the molecule together but gives the molecule structure and strength. The distance separating the two strands that make up the DNA structure is about 1 nm, while the distance separating the individual atoms within each base is about 0.3 nm. One might wonder why electrostatic forces do not play a larger role in biology than they do if we have so many charged molecules. The reason is that the electrostatic force is "diluted" due to screening between molecules. This is due to the presence of other charges in the cell.

3.4.3.2 Polarity of Water Molecules

The best example of this charge screening is the water molecule, represented as H2 O . Water is a strongly polar molecule. Its 10 electrons (8 from the oxygen atom and 2 from the two hydrogen atoms) tend to remain closer to the oxygen nucleus than the hydrogen nuclei. This creates two centers of equal and opposite charges—what is called a dipole. The magnitude of the dipole is called the dipole moment. These two centers of charge will terminate some of the electric field lines coming from a free charge, as on a DNA molecule. This results in a reduction in the strength of the Coulomb interaction. One might say that screening makes the Coulomb force a short range force rather than long range. Other ions of importance in biology that can reduce or screen Coulomb interactions are Na^+ , and K^+ , and Cl^- . These ions are located both inside and outside of living cells. The movement of these ions through cell membranes is crucial to the motion of nerve impulses through nerve axons. Recent studies of electrostatics in biology seem to show that electric fields in cells can be extended over larger distances, in spite of screening, by "microtubules" within the cell. These microtubules are hollow tubes composed of proteins that guide the movement of chromosomes when cells divide, the motion of other organisms within the cell, and provide mechanisms for motion of some cells (as motors).

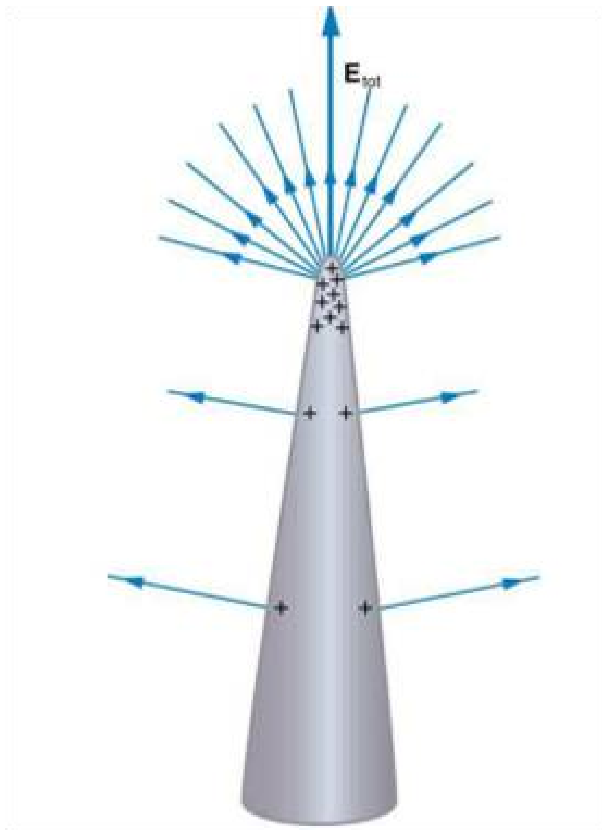
3.4.3.3 Earth's Electric Field

A near uniform electric field of approximately 150 N/C, directed downward, surrounds Earth, with the magnitude increasing slightly as we get closer to the surface. What causes the electric field? At around 100 km above the surface of Earth we have a layer of charged particles, called the ionosphere. The ionosphere is responsible for a range of phenomena including the electric field surrounding Earth. In fair weather the ionosphere is positive and the Earth largely negative, maintaining the electric field. In storm conditions clouds form and localized electric fields can be larger and reversed in direction (Figure 18.34(b)). The exact charge distributions depend on the local conditions, and variations are possible. If the electric field is sufficiently large, the insulating properties of the surrounding material break down and it becomes conducting. For air this occurs at around 3×10^6 N/C. Air ionizes ions and electrons recombine, and we get discharge in the form of lightning sparks and corona discharge.

Lightning

3.4.3.4 Applications of Conductors

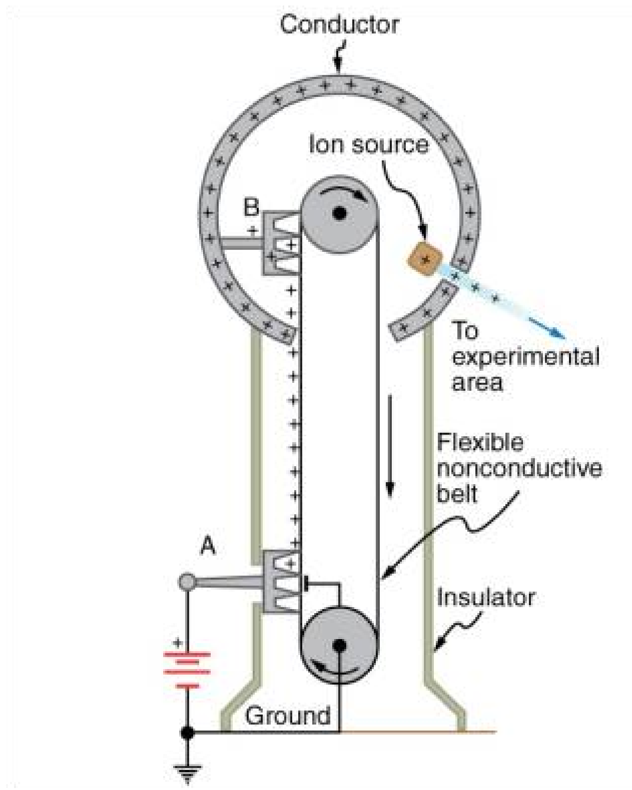
On a very sharply curved surface, such as shown in Figure , the charges are so concentrated at the point that the resulting electric field can be great enough to remove them from the surface. This can be useful. Lightning rods work best when they are most pointed. The large charges created in storm clouds induce an opposite charge on a building that can result in a lightning bolt hitting the building. The induced charge is bled away continually by a lightning rod, preventing the more dramatic lightning strike. Of course, we sometimes wish to prevent the transfer of charge rather than to facilitate it. In that case, the conductor should be very smooth and have as large a radius of curvature as possible. (See Figure 18.37.) Smooth surfaces are used on high-voltage transmission lines, for example, to avoid leakage of charge into the air. Another device that makes use of some of these principles is a Faraday cage. This is a metal shield that encloses a volume. All electrical charges will reside on the outside surface of this shield, and there will be no electrical field inside. A Faraday cage is used to prohibit stray electrical fields in the environment from interfering with sensitive measurements, such as the electrical signals inside a nerve cell. During electrical storms if you are driving a car, it is best to stay inside the car as its metal body acts as a Faraday cage with zero electrical field inside. If in the vicinity of a lightning strike, its effect is felt on the outside of the car and the inside is unaffected, provided you remain totally inside. This is also true if an active ("hot") electrical wire was broken (in a storm or an accident) and fell on your car.



A Sharp Conductor and its Electric field

3.4.3.5 The Van de Graaff Generator

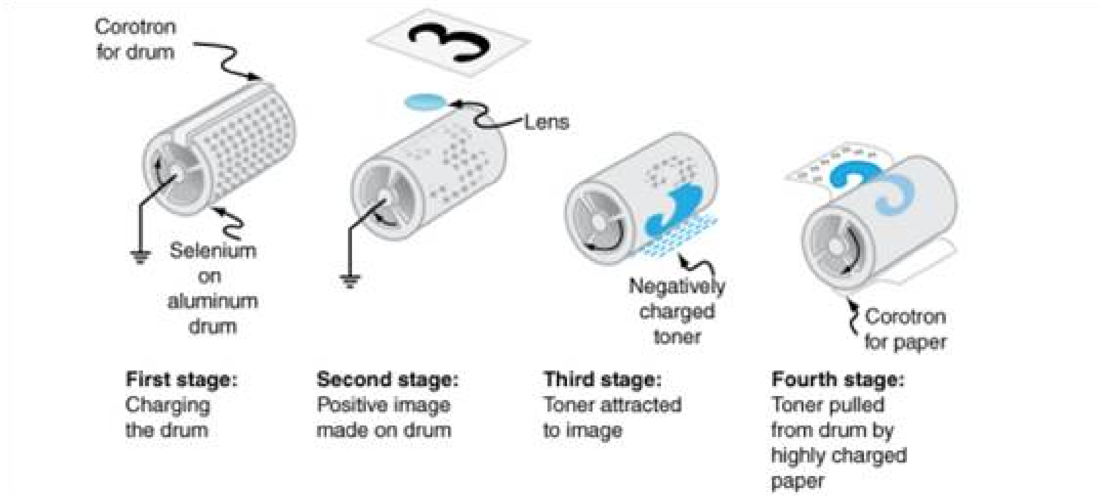
Van de Graaff generators (or Van de Graaffs) are not only spectacular devices used to demonstrate high voltage due to static electricity—they are also used for serious research. The first was built by Robert Van de Graaff in 1931 (based on original suggestions by Lord Kelvin) for use in nuclear physics research. Figure shows a schematic of a large research version. Van de Graaffs utilize both smooth and pointed surfaces, and conductors and insulators to generate large static charges and, hence, large voltages. A very large excess charge can be deposited on the sphere, because it moves quickly to the outer surface. Practical limits arise because the large electric fields polarize and eventually ionize surrounding materials, creating free charges that neutralize excess charge or allow it to escape. Nevertheless, voltages of 15 million volts are well within practical limits.



Van de Grafe generator

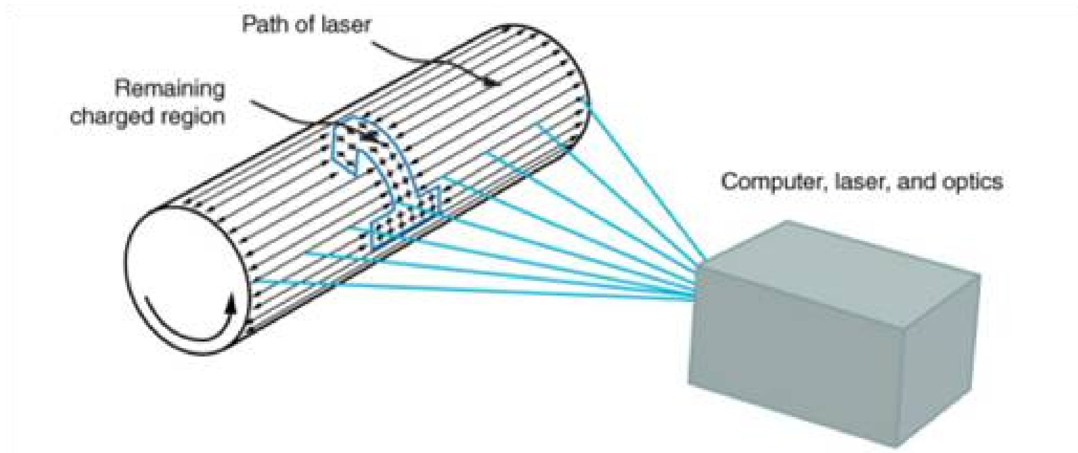
3.4.3.6 Xerography

Most copy machines use an electrostatic process called xerography—a word coined from the Greek words xeros for dry and graphos for writing. The heart of the process is shown in simplified form in Figure . A selenium-coated aluminum drum is sprayed with positive charge from points on a device called a corotron. Selenium is a substance with an interesting property—it is a photoconductor. That is, selenium is an insulator when in the dark and a conductor when exposed to light. In the first stage of the xerography process, the conducting aluminum drum is grounded so that a negative charge is induced under the thin layer of uniformly positively charged selenium. In the second stage, the surface of the drum is exposed to the image of whatever is to be copied. Where the image is light, the selenium becomes conducting, and the positive charge is neutralized. In dark areas, the positive charge remains, and so the image has been transferred to the drum. The third stage takes a dry black powder, called toner, and sprays it with a negative charge so that it will be attracted to the positive regions of the drum. Next, a blank piece of paper is given a greater positive charge than on the drum so that it will pull the toner from the drum. Finally, the paper and electrostatically held toner are passed through heated pressure rollers, which melt and permanently adhere the toner within the fibers of the paper.



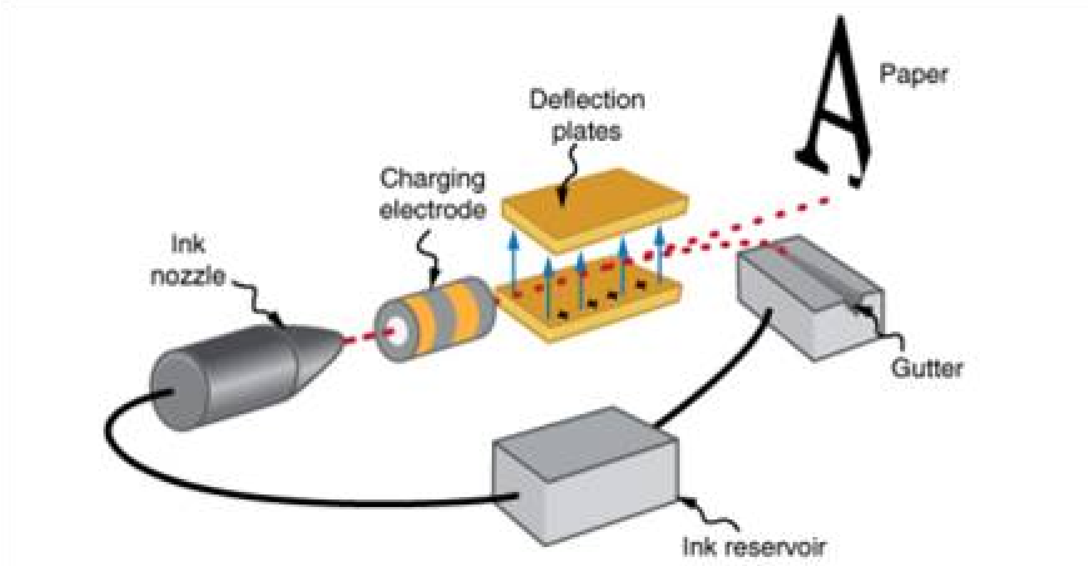
3.4.3.7 Laser Printers

Laser printers use the xerographic process to make high-quality images on paper, employing a laser to produce an image on the photoconducting drum as shown in Figure . In its most common application, the laser printer receives output from a computer, and it can achieve high-quality output because of the precision with which laser light can be controlled. Many laser printers do significant information processing, such as making sophisticated letters or fonts, and may contain a computer more powerful than the one giving them the raw data to be printed.



3.4.3.8 Ink Jet Printers and Electrostatic Painting

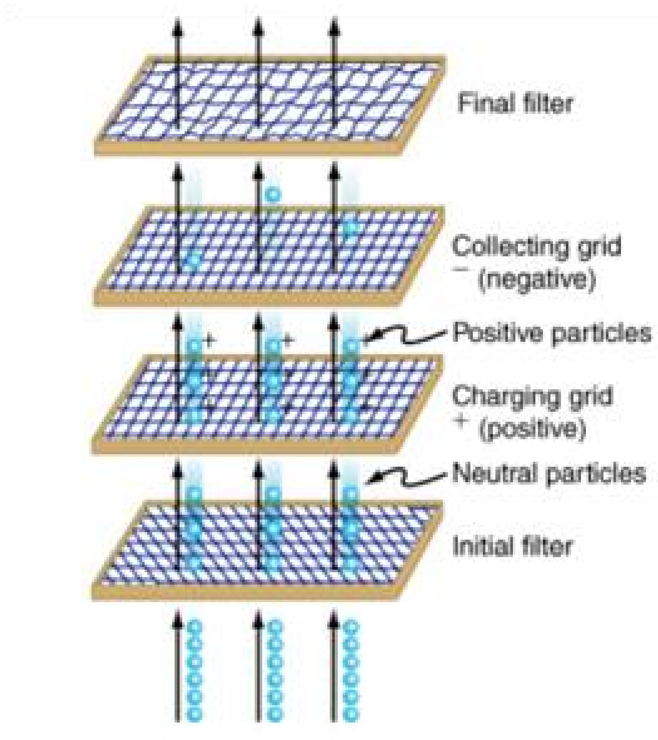
The ink jet printer, commonly used to print computer-generated text and graphics, also employs electrostatics. A nozzle makes a fine spray of tiny ink droplets, which are then given an electrostatic charge. Once charged, the droplets can be directed, using pairs of charged plates, with great precision to form letters and images on paper. Ink jet printers can produce color images by using a black jet and three other jets with primary colors, usually cyan, magenta, and yellow, much as a color television produces color. (This is more difficult with xerography, requiring multiple drums and toners.)



Ink jet Printer

3.4.3.9 Smoke Precipitators and Electrostatic Air Cleaning

Another important application of electrostatics is found in air cleaners, both large and small. The electrostatic part of the process places excess (usually positive) charge on smoke, dust, pollen, and other particles in the air and then passes the air through an oppositely charged grid that attracts and retains the charged particles. Large electrostatic precipitators are used industrially to remove over 99% of the particles from stack gas emissions associated with the burning of coal and oil. Home precipitators, often in conjunction with the home heating and air conditioning system, are very effective in removing polluting particles, irritants, and allergens.



Electrostatic Precipitators

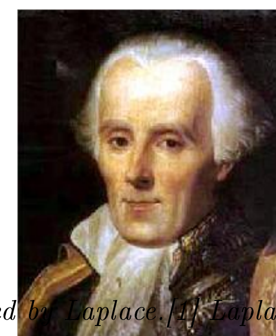
Unit-II

Conductors And Dipole:

Laplace's and Poison's equations – Solution of Laplace's equation in one variable. Electric dipole – Dipole moment – potential and EFI due to an electric dipole – Torque on an Electric dipole in an electric field – Behavior of conductors in an electric field – Conductors and Insulators.

Chapter 4

POISSON'S AND LAPLACE'S EQUATIONS:



Pierre-Simon, marquis de Laplace(23 March 1749 – 5 March 1827) *was a French mathematician and astronomer whose work was pivotal to the development of mathematical astronomy and statistics. He summarized and extended the work of his predecessors in his five-volume Mécanique Céleste (Celestial Mechanics) (1799–1825). This work translated the geometric study of classical mechanics to one based on calculus, opening up a broader range of problems. In statistics, the so-called Bayesian interpretation of probability was mainly developed by Laplace.[1] Laplace formulated Laplace's equation, and pioneered the Laplace transform which appears in many branches of mathematical physics, a field that he took a leading role in forming.The Laplacian differential operator, widely used in mathematics, is also named after him. He restated and developed the nebular hypothesis of the origin of the solar system and was one of the first scientists to postulate the existence of black holes and the notion of gravitational collapse. Laplace is remembered as one of the greatest scientists of all time. Sometimes referred to as the French Newton or Newton of France, he possessed a phenomenal natural mathematical faculty superior to that of any of his contemporaries.[2] Laplace became a count of the First French Empire in 1806 and was named a marquis in 1817, after the Bourbon Restoration.*

Siméon Denis Poisson (21 June 1781 – 25 April 1840), *was a French mathematician, geometer, and physicist. He obtained many important results, but within the elite Académie des Sciences he also was the final leading opponent of the wave theory of light and was proven wrong on that matter by Augustin-Jean Fresnel.*

4.1 DERIVATION OF LAPLACE'S AND POISSON'S EQUATIONS:

The Poisson's equation can be derived from the point form of Gauss's law

$$\begin{aligned}
\nabla \bullet D &= \rho_v \\
D &= \epsilon E \\
E &= -\nabla V \\
\nabla \bullet D &= \nabla \bullet (\epsilon E) = -\nabla \bullet (\epsilon \nabla V) = \rho_v \\
\nabla \bullet \nabla V &= -\frac{\rho_v}{\epsilon}
\end{aligned}$$

In the above equation ϵ is a constant.

The equation is known as the Poisson's equation a and in Cartesian coordinates it is given as

$$\nabla \bullet \nabla V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \tag{4.1}$$

The operation $\nabla \bullet \nabla$ is abbreviated as ∇^2 and we have

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \tag{4.2}$$

in cartesian coordinates.

If $\rho_v = 0$, indicating zero volume charge density , but allowing point charges, line charges, and surface charge density to exist at singular locations as sources of the field, then

$$\nabla^2 V = 0 \tag{4.3}$$

which is Laplace's equation. The ∇^2 operation is called the Laplacian of V . In cartesian coordinates

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \tag{4.4}$$

In cylindrical coordinates

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2} \tag{4.5}$$

In spherical coordinates

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} \tag{4.6}$$

Laplace's equation is all embracing , for, applying as it does where volume charge density is zero, it states that every conceivable configuration of electrodes or conductors produces a field for which $\nabla^2 V = 0$. All these fields are different , with different potential values and different spatial rates of change, yet for each of them $\nabla^2 V = 0$. Since every field (if $\rho_v = 0$) satisfies Laplace,s equation , how can we expect to reverse the procedure and use Laplace's equation to find one specific field in which we happen to to have an interest? Obviously more information is required, and we shall find that we must solve Laplace's equation subject to certain boundary conditions.

Every physical problem must contain at least one conducting boundary and usually contains two or more . The potentials on these boundaries are assigned values, perhaps $V_0, V_1, \cdots$ or perhaps numerical values. These definite equipotential surfaces will provide the boundary conditions for the type of problem to be solved . In other types of problems , the boundary conditions take the form of specified values of E on an enclosing surface, or a mixture of known values of V and E . It is necessary to show that if our answer satisfies the Laplace's equation and also satisfies the boundary conditions, then it is the only possible answer.

4.1.1 UNIQUENESS THEOREM:

Let us assume that we have two solutions of Laplace's equation, V_1 and V_2 , both general functions of the coordinates used. Therefore

$$\nabla^2 V_1 = 0 \tag{4.7}$$

and

$$\nabla^2 V_2 = 0 \tag{4.8}$$

from which

$$\nabla^2 (V_1 - V_2) = 0 \tag{4.9}$$

Each solution must also satisfy the boundary conditions , and if we represent the given potential values on the boundaries by V_b , then the value of V_1 on the boundary V_{1b} and the value of V_2 on the boundary V_{2b} must both be identical to V_b

$$V_{1b} = V_{2b} = V_b \tag{4.10}$$

or

$$V_{1b} - V_{2b} = 0 \tag{4.11}$$

Using the vector identity which will hold for for any scalar $V$and vector D

$$\nabla \bullet (VD) = V(\nabla \bullet D) + D \bullet (\nabla V) \tag{4.12}$$

For the present we will assume that $V = V_1 - V_2$ is the scalar and $\nabla(V_1 - V_2)$ as the vector, giving

$$\nabla\bullet[(V_1 - V_2)\,\nabla\,(V_1 - V_2)] = (V_1 - V_2)\,[\nabla \bullet \nabla\,(V_! - V_2)]+\nabla\,(V_! - V_2)\bullet\nabla\,(V_1 - V_2) \tag{4.13}$$

which we will integrate throughout the volume enclosed by the bounding surfaces specified

$$\int\limits_{vol} \nabla\bullet[(V_1 - V_2)\,\nabla\,(V_1 - V_2)]\,dv = \int\limits_{vol} (V_1 - V_2)\,[\nabla \bullet \nabla\,(V_! - V_2)]+\nabla\,(V_! - V_2)\bullet\nabla\,(V_1 \tag{4.14}$$

The divergence theorem allows us to replace the volume integral on the left side of the equation by the closed surface integral over the surface surrounding the volume. this surface consists of the boundaries already specified on which $V_{1b} = V_{2b}$, and therefore

$$\int\limits_{vol} \nabla\bullet[(V_1 - V_2)\,\nabla\,(V_1 - V_2)]\,dv = \oint\limits_{s} [(V_{1b} - V_{2b})\nabla\,(V_{1b} - V_{2b})]\bullet ds = 0 \tag{4.15}$$

One of the factors of the first integral on the right side is $\nabla\bullet\nabla(V_1 - V_2)$or $\nabla^2(V_1 - V_2)$ which is zero by hypothesis , and therefore that integral is zero. Hence the volume integral must be zero.

$$\int\limits_{vol} [\nabla\,(V_1 - V_2)]^2\,dv = 0 \tag{4.16}$$

There are in general two reasons why an integral may be zero: either the integrand (the quantity under the integral sign) is everywhere zero, or the integrand is positive in regions and negative in others, and the contributions cancel algebraically. In this case the first reason must hold good because $[\nabla(V_1 - V_2)]^2$ can not be negative. Therefore

$$[\nabla\,(V_1 - V_2)]^2 = 0 \tag{4.17}$$

and

$$\nabla(V_1 - V_2) = 0 \tag{4.18}$$

Finally, if the gradient of $V_1 - V_2$ is everywhere zero , then $V_1 - V_2$ can not change with any coordinates and

$$V_! - V_2 = \text{Constant} \tag{4.19}$$

If we can show that t5his constant is zero, we shall have accomplished our proof . The constant is easily evaluated by considering a point on the boundary . Here $V_1 - V_2 = V_{1b} - V_{2b} = 0$, and we see that the constant is indeed zero, and therefore

$$V_1 = V_2 \tag{4.20}$$

giving two identical solutions.

The uniqueness theorem also is applicable to Poisson's equation, for if $\nabla^2 V_1 = -\frac{\rho_v}{\epsilon}$ and $\nabla^2 V_2 = -\frac{\rho_v}{\epsilon}$, then , $\nabla^2(V_1 - V_2) = 0$ as before. Boundary conditions still require that $V_{1b} - V_{2b} = 0$, and the proof is identical from this point.

4.1.2 EXAMPLES:

Several methods have been developed for solving the second order partial differential equation known as Laplace's equation . The first and simplest method is that of direct integration, and we shall use this technique to work several examples in various coordinate systems.

The method of direct integration is applicable only to problems which are one dimensional or in which the potential field is a function of only one of the three coordinates. Since we are working with only three coordinate systems, it might seem that there are nine problems to be solved, but a little reflection will show that a field which varies only with x is fundamentally the same as with y.Rotating the physical problem a quarter turn is no change. Actually, there are only five problems to be solved , one in cartesian coordinates, two in cylindrical coordinates, and two in spherical coordinates.

Example:

let us assume that $V$is a function only of x and worry later about which physical problem we are solving when we have a need for boundary conditions. Laplace's equation reduces to

$$\frac{\partial^2 V}{\partial x^2} = 0 \tag{4.21}$$

and the partial derivative may be replaced by an ordinary derivative , since V is not a function of y or z ,

$$\frac{d^2 V}{dx^2} = 0 \tag{4.22}$$

we integrate twice , obtaining

$$\frac{dV}{dx} = A \ , \ \ V = Ax + B \tag{4.23}$$

where A and B are constants of integration. These constants can be determined only from the boundary conditions. Since the field varies only with x and is not a function of $y$and z , then V is a constant if x is a constant or in other words, the equipotential surfaces are described by setting x constant. These surfaces are parallel planes normal to the $x$axis. the field is thus of a parallel plate capacitor,and as soon as we specify the potential on any two planes , we may evaluate our constants of integration.

Let $V = V_1$ at $x = x_1$and $V = V_2$ at $x = x_2$. These values are then substituted giving

$$\begin{aligned} V_1 &= Ax_1 + B \\ V_2 &= Ax_2 + B \\ A &= \frac{V_! - V_2}{x_1 - x_2} \quad B = \frac{V_2 x_1 - V_1 x_2}{x_1 - x_2} \end{aligned}$$

$$V = \frac{V_1(x - x_2) - V_2(x - x_1)}{x_1 - x_2} \tag{4.24}$$

The general solution is

$$V = mx + b \tag{4.25}$$

a straight line equation. If $V = 4$ at $x = 1$ and $V = 0$ at $x = 5$, then $m = -1$ and $b = 5$. Then

$$V = -x + 5 \tag{4.26}$$

The above solution has two properties

1. $V(x)$ is the average of $V(x + R)$ and $V(x - R)$

$$V(x) = \frac{1}{2}[V(x + R) + V(x - R)] \tag{4.27}$$

Laplace's equation is a kind of averaging instruction. It tells you to assign to the point x , the average of the value to the left and to the right of x .

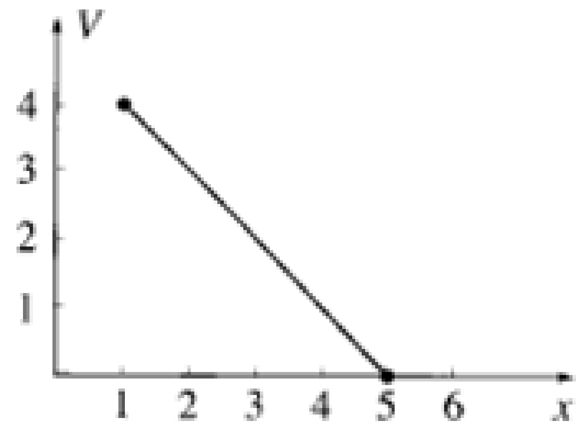


Figure 4.1: Graph Of $V = -x + 5$

2. Laplace's equation tolerates no local maxima . Extreme values of V must occur at the end points . This is a consequence of property 1 .

A simple answer would have been obtained by choosing simpler boundary conditions. If we had fixed $V = 0$ at $x = 0$ and $V = V_0$ at $x = d$, then

$$\begin{aligned} A &= \frac{V_0}{d} \\ B &= 0 \end{aligned}$$

and

$$V = \frac{V_0 x}{d} \tag{4.28}$$

Suppose our primary aim is to find the capacitance of a parallel plate capacitor. We have solved Laplace's equation , obtaining the two constants A and B . We are not interested in the potential field itself , but only in the capacitance, then we may continue successfully with A and b or we may simplify the algebra by little foresight. Capacitance is given by the ratio of charge to potential difference , so we may chose now the potential difference as V_0 , which is equivalent to one boundary condition, and then choose whatever second boundary condition seems to help the form of the equation the most. This is what we did while choosing the second set of boundary conditions. The potential difference is fixed as V_0 by choosing the potential of plate zero and the other V_0; the location of these plates was made as simple as possible by letting $V = 0$ at $x = 0$.

We still need the total charge on either plate before the capacitance can be found. The necessary steps are these

1. Given V, use $E = -\nabla V$ to find E
2. Use $D = \epsilon E$ to find D
3. Evaluate D at either of the plates , $D = D_s = D_N a_N$
4. Recognize that $\rho_s = D_N$
5. Find Q by surface integration over the capacitor plate , $Q = \int_s \rho_s ds$

Here we have

$$\begin{aligned}
V &= V_0 \frac{x}{d} \\
E &= -\frac{V_0}{d} a_x \\
D &= -\epsilon \frac{V_0}{d} a_x \\
D_s &= D|_{x=0} = -\epsilon \frac{V_0}{d} a_x \\
a_N &= a_x \\
D_N &= -\epsilon \frac{V_0}{d} = \rho_s \\
Q &= \int_s -\frac{\epsilon V_0}{d} ds = -\epsilon \frac{V_0 S}{d}
\end{aligned}$$

and the capacitance is

$$C = \frac{|Q|}{V_0} = \frac{\epsilon S}{d} \tag{4.29}$$

4.2 Electric Dipole

4.2.1 POTENTIAL AND ELECTRIC FIELD OF A DIPOLE:

An electric dipole is formed by two point charges of equal magnitude and opposite sign $(+Q, -Q)$ separated by a short distance d. The potential at the point P due to the electric dipole is found using superposition.

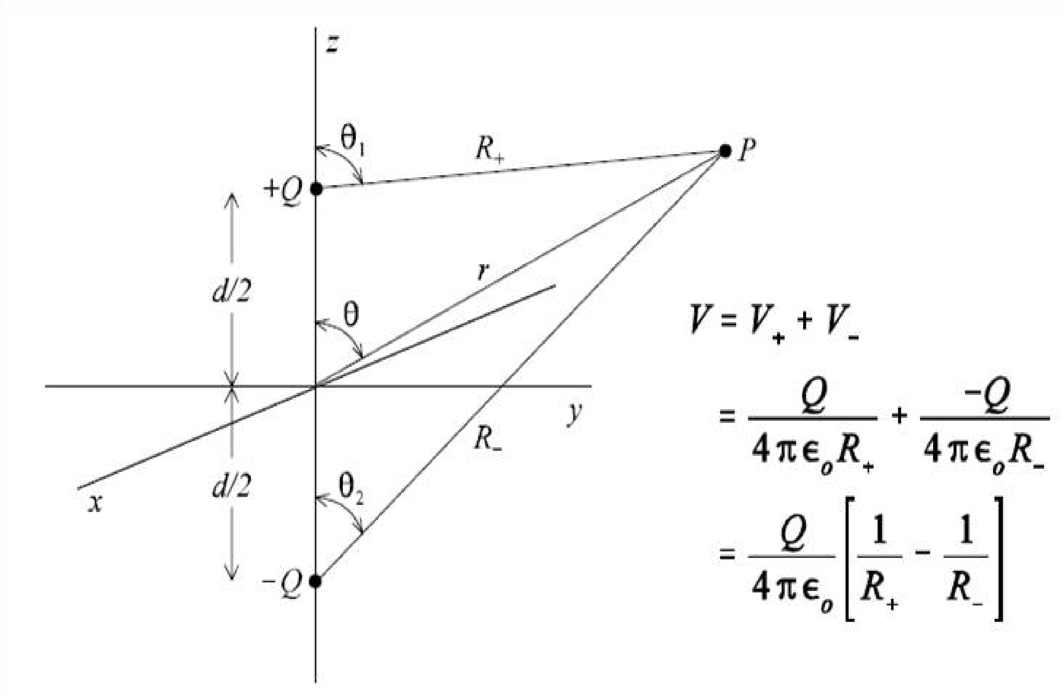


Figure 4.2: Dipole

If the field point P is moved a large distance from the electric dipole (in what is called the far field, $r \gg d$ the lines connecting the two charges and the coordinate origin with the field point become nearly parallel.

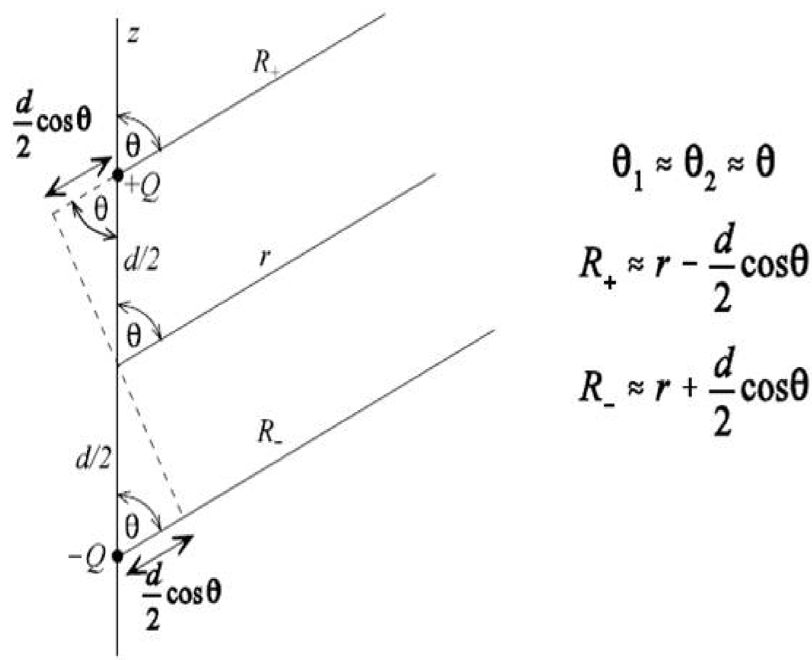


Figure 4.3: Far field approximation

$$V \approx \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r-\frac{d}{2}\cos\theta}-\frac{1}{r+\frac{d}{2}\cos\theta}\right]$$

$$V \approx \frac{Q}{4\pi\epsilon_0}\left[\frac{\left(r+\frac{d}{2}\cos\theta\right)-\left(r-\frac{d}{2}\cos\theta\right)}{\left(r^2-\frac{d^2}{4}\cos^2\theta\right)}\right]$$

as $r \gg d$ the far field is

$$V = \frac{Qd\cos\theta}{4\pi\epsilon_0 r^2} \tag{4.30}$$

The electric field produced by the electric dipole is found by taking the gradient of the potential.

$$\begin{aligned}
V = -\nabla E &= -\left[\frac{\partial V}{\partial r}a_r + \frac{1}{r}\frac{\partial V}{\partial \theta}a_\theta\right] \\
&= -\frac{Qd}{4\pi\epsilon_0}\left[\cos\theta\frac{\partial}{\partial r}\left(\frac{1}{r^2}\right)a_r + \frac{1}{r^3}\frac{\partial}{\partial \theta}(\cos\theta)a_\theta\right] \\
&= -\frac{Qd}{4\pi\epsilon_0}\left[\cos\theta\left(-\frac{2}{r^3}\right)a_r + \frac{1}{r^3}(-\sin\theta)a_\theta\right] \\
&= \frac{Qd}{4\pi\epsilon_0 r^3}\left[2\cos\theta a_r + (\sin\theta)a_\theta\right]
\end{aligned}$$

If the vector dipole moment is defined as

$$P = pa_p = Qda_p \tag{4.31}$$

where a_p points from $+Q$ to $-Q$. The dipole potential and electric field may be written as

$$\boxed{\begin{aligned}
V &= \frac{Qd\cos\theta}{4\pi\epsilon_0 r^2} = \frac{P \bullet a_r}{4\pi\epsilon_0 r^2} \\
E &= \frac{P}{4\pi\epsilon_0 r^3}\left[2\cos\theta a_r + \sin\theta a_\theta\right]
\end{aligned}}$$

Note that the potential and electric field of the electric dipole decay faster than those of a point charge. For an arbitrarily located, arbitrarily oriented dipole, the potential can be written as

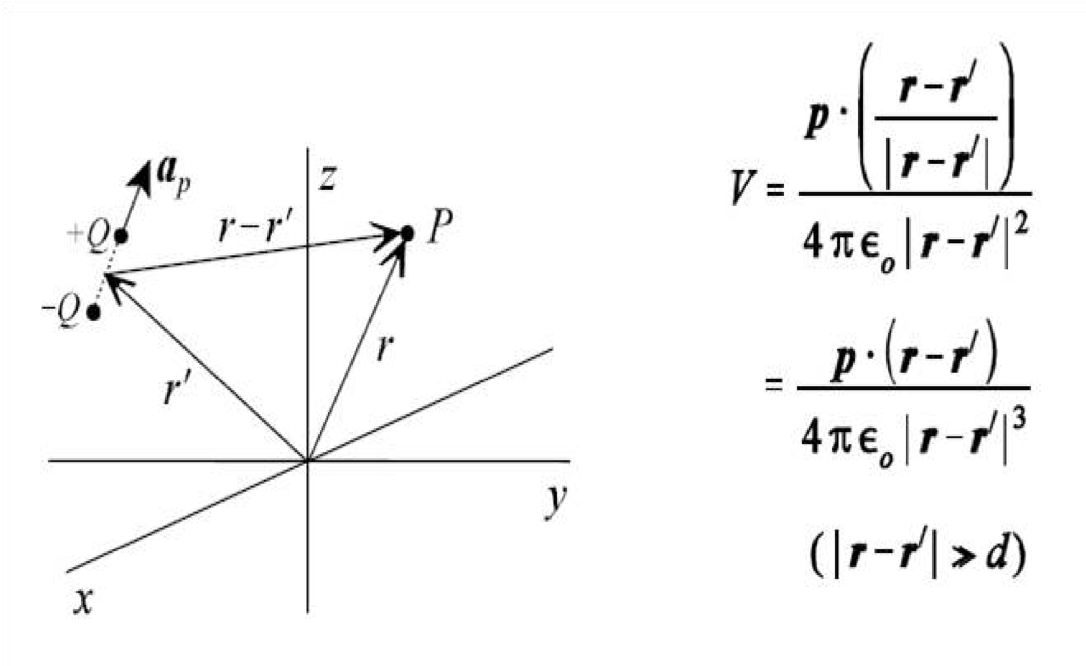
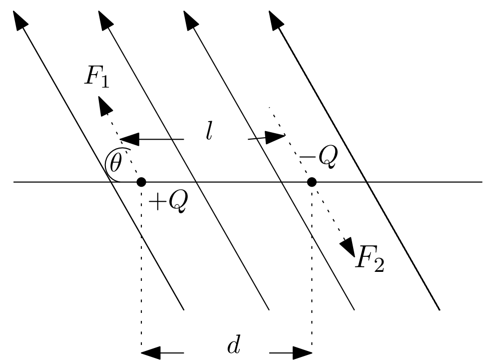


Figure 4.4: Arbitrarily placed dipole

$$
\begin{aligned}
V =& \frac{P \bullet \left(\frac{r - r'}{|r - r'|} \right)}{4\pi\epsilon_0 \left| r - r' \right|^2} \\
=& \frac{P \bullet (r - r')}{4\pi\epsilon_0 \left| r - r' \right|^3} \\
|r - r'| >>& d
\end{aligned}
$$

4.2.2 Torque On A Dipole In an Electric Field



The dipole moment is given by

$$p = Qd \tag{4.32}$$

It is a vector directed from the negative to positive charge forming the dipole. The potential at any point because of the dipole is given by

$$V = \frac{p \bullet a_r}{4\pi\epsilon_0 r^2} \tag{4.33}$$

What happens when a dipole is placed in a uniform electric field? Will it experience a force ? There are two charges Q and $-Q$ forming the dipole, each of which experiences a force equal in magnitude to QE but oppositely directed, with the result that the dipole experiences no tanslational force , as forces F_1 and F_2 neutralize is other, but these forces form a couple, whose torque is equal in magnitude to force X length of the arm of the couple.

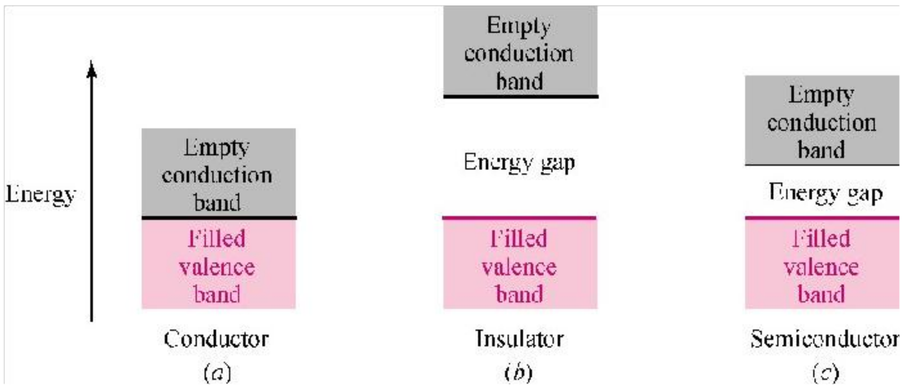
$$\begin{aligned}
T &= (QE)\, l = QEd \sin\theta \\
T &= Qd\,(E \sin\theta) \\
T &= pE \sin\theta \\
T &= p \times E
\end{aligned}$$

Although a dipole in a uniform field does not experience a translational force , it does experience a torque tending to align the dipole axis with the field direction.

4.2.3 Conductors, Semiconductors, and Insulators

Electrons surrounding the positive atomic nucleus are described in terms of the total energy of the electron with respect to a zero reference level for an electron at an infinite distance from the nucleus. The total energy is the sum of the kinetic and potential energies , and since energy must be given to an electron to pull it away from the nucleus, the energy of every electron in the atom is a negative quantity. It is convenient to associate these energy levels , or energy states , are permissible in a given atom, and an electron must therefore absorb r emit discrete amounts of energy or quanta in passing from one level to another.

In a crystalline solid , such as a metal or a diamond, atoms are packed closely together, many more permissible energy levels are available because of the interaction forces between adjacent atoms. it can be observed that energies which may be possessed by electrons are grouped into broad ranges or "bands" , each band consisting of very numerous , closely spaced, discrete levels. At a temperature of absolute zero, the normal solid also has every level occupied, starting with the lowest and proceeding in order until all the electrons are located. The electrons with the highest (least negative) energy levels , the valance electrons are located in the valance band. If there are permissible higher energy levels in the valance band , or if the valance band merges smoothly into a conduction band, then the additional kinetic energy may be given to the valence electrons by an external field , resulting in an electron flow. The solid is called a conductor. The filled valance band and the unfilled conduction band are shown for conductor



If however the electron with the greatest energy occupies the top level in the valance band and a gap exists between the valance band and the conduction band , then the electron cannot accept additional energy in small amounts and the material is an insulator. The band structure is indicated in the above figure. Note that if a relatively large amount of energy can be transferred to the electron , it may be sufficiently excited , to jump the gap into the next band , where conduction occur easily. here the insulator breaks down.

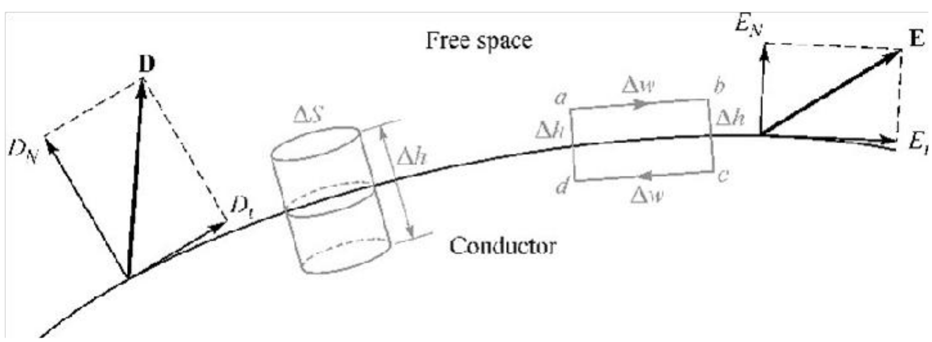
An intermediate condition occurs when only a small " forbidden region separates the two bands as indicated in the figure. Small amounts of energy in the form of heat, or an electric field may raise the energy of the electrostatic the top of the filled band and provide the basis for conduction. These materials are insulators which display many of the properties of the conductors and are called semiconductors.

4.2.4 Conductor free space boundary

What happens when suddenly the charge distribution is unbalanced within a conducting material/ Let us suppose that there suddenly appear a number of electrons in the interior of the conductor. The electrical field setup by these electrons are not countered by any positive charges, and the electrons begin to accelerate away from each other . this continues until the electrons reach the surface of the conductor . Here the outward progress of the electrons is stopped , for the material surrounding the conductor is an insulator, not possessing a conduction band. No charge will remain within the conductor Hence the final result within a conductor is zero charge density , and a surface charge density resides on the exterior surface.

also for static conditions in which no current may flow , the electric field intensity within the conductor is zero.

So for electrostatics , no charge and no electric field may exist at any point within a conductor. charge may appear on the surface as surface charge density. Ther will be a field external to the conductor , and this field can be decomposed into two components , one tangential and one normal to the conductor surface.



the tangential component is seen to be zero . If it were not zero , a tangential force would be applied to the elements of the surface, resulting in their motion and non-static conditions . Since static conditions are assumed , the tangential electric field intensity and electrical flux density are zero.

$$\oint E \bullet dl = 0$$

$$E_t \Delta w = 0, E_t = 0$$

$$\oint_s D \bullet ds = Q = \rho_s \Delta s$$

$$D_N \Delta s = \rho_s \Delta s$$

$$D_N = \rho_s \ , D_t = E_t = 0$$

In summary

1. The static electric field inside a conductor is zero
2. The static electric field intensity at the surface of a conductor is every where normal to the surface of the conductor
3. The conductor surface is an equipotential surface

Unit-III

Dielectric And Capacitance:

Electric field inside a dielectric material – polarization – Dielectric – Conductor and Dielectric – Dielectric boundary conditions, Capacitance – Capacitance of parallel plate and spherical and coaxial capacitors with composite dielectrics– Energy stored and energy density in a static electric field – Current density – conduction and Convection current densities – Ohm's law in point form – Equation of continuity

Chapter 5

Polarization

Georg Simon Ohm (16 March 1789 – 6 July 1854) was a Bavarian (German) physicist and mathematician. As a high school teacher, Ohm began his research with the new electrochemical cell, invented by Italian scientist Alessandro Volta. Using equipment of his own creation, Ohm found that there is a direct proportionality between the potential difference (voltage) applied across a conductor and the resultant electric current. This relationship is known as Ohm's law.

Conductors are characterized by an abundance of conduction, or free electrons which can move. charges in a dielectric are not able to move about freely. They are bound by finite forces. An external electric field may displace them.

To understand, consider an atom consisting of a charge Q and a charge $-Q$. The same picture can be used to describe a dielectric molecule. Since there is an equal amount of positive and negative charge the molecule is neutral electrically. When an electric field is applied , the positive and negative charges are displaced in space slightly. A dipole results and the dielectric is polarized. In the polarized state , the electron cloud is distorted by the applied electric field E . This distorted charge distribution is equal, by the principle of superposition, to the original distribution plus a dipole whose moment is $p = Qd$, where d is the distance vector from $-Q$ and Q . If there are N dipoles in a volume Δv the total dipole moment is

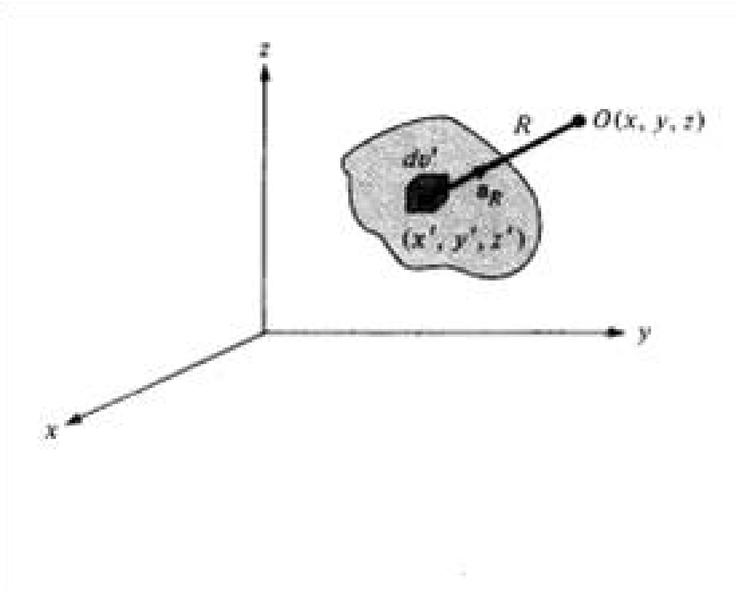
$$Q_1 d_1 + Q_2 d_2 + \cdots\cdots + Q_N d_N = \sum_{k=1}^{N} Q_k d_k \tag{5.1}$$

Polarization is defined as

$$P = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{N} Q_k d_k}{\Delta v} = \frac{\text{dipole moment}}{\text{unit volume}} \tag{5.2}$$

This type of dielectric is called non-polar. They do not posses dipoles until an electric field is applied . Ex: Hydrogen, nitrogen, and the rare gases.

some dielectrics have built in permanent dipoles which are randomly oriented, and are said to be polar. Ex: water, sulfurdioxide, hydrochloric acid etc,. When an electric field is applied , the dipole experiences a torque, tending to align its dipole moment parallel to E . The potential at any external point dV at O is



$$
\begin{aligned}
dV &= \frac{P \bullet a_R dv'}{4\pi\epsilon_0 R^2} \\
R^2 &= (x-x')^2 + (y-y')^2 + (z-z')^2 \\
\nabla\left(\frac{1}{R}\right) &= \frac{a_R}{R^2} \\
\frac{P \bullet a_R}{R^2} &= P \bullet \nabla'\left(\frac{1}{R}\right) \\
\nabla' \bullet (f\,A) &= f\nabla' \bullet A + A \bullet \nabla' f \\
\frac{P \bullet a_R}{R^2} &= \nabla' \bullet \left(\frac{P}{R}\right) - \frac{\nabla' \bullet P}{R}
\end{aligned}
$$

Substituting this and integrating over the entire volume

$$V = \int\limits_{v'} \frac{1}{4\pi\epsilon_0}\left[\nabla' \bullet \left(\frac{P}{R}\right) - \frac{\nabla' \bullet P}{R}\right] dv' \tag{5.3}$$

Applying divergence theorem to the first term

$$V = \oint_{s'} \frac{P \bullet a'_n}{4\pi\epsilon_0 R} ds' + \int -\frac{\nabla' \bullet P}{4\pi\epsilon_0 R} dv' \tag{5.4}$$

a'_n is the outward unit normal to the surface ds' . The two terms show that the potential is because of a surface charge distribution and a volume charge distribution.

$$\rho_{\rho s} = P \bullet a_n \longrightarrow \text{(polarization) bound surface charge}$$
$$\rho_{\rho v} = -\nabla' \bullet P \longrightarrow \text{Volume charge distribution}$$

If ρ_v is the free volume charge density , the total volume charge density is

$$\begin{aligned} \rho_T &= \rho_v + \rho_{\rho v} = \nabla \bullet \epsilon_0 E \\ \rho_v &= \nabla \bullet \epsilon_0 E - \rho_{\rho v} = \nabla \bullet \epsilon_0 E + \nabla \bullet P \\ &= \nabla \bullet (\epsilon_0 E + P) = \nabla \bullet D \end{aligned}$$

Hence

$$D = \epsilon_0 E + P \tag{5.5}$$

The polarization P and the electric field E are linearly related for most materials and is given by

$$P = \chi_e \epsilon_0 E \tag{5.6}$$

so

$$D = \epsilon_0 E + P = \epsilon_0 E + \chi_e \epsilon_0 E = \epsilon_0 (1 + \chi_e) E$$

χ_e is called the electric susceptibility and

$$1 + \chi_e = \epsilon_R \tag{5.7}$$

so

$$D = \epsilon_0 \epsilon_R E$$
$$D = \epsilon E$$

where

$$\epsilon_R = \text{relative permitivity}$$
$$\epsilon_0 = \text{dielectric constant of the medium}$$

The dielectric constant (or relative permittivity) ϵ_r, is the ratio of the permittivity of the dielectric to that of free space.

It should also be noticed that $\epsilon_r$and χ_e are dimensionless whereas ϵ and ϵ_0 are in farads/meter. The approximate values of the dielectric constants of some common materials as are given in Table. The values given in Table are for static or low frequency (<1000 Hz) fields; the values may change at high frequencies. Note from the table that ϵ_r is always greater or equal to unity. For free space and non dielectric mate-rials (such as metals) $\epsilon_r = 1$. The theory of dielectrics we have discussed so far assumes ideal dielectrics. Practically speaking, no dielectric is ideal. When the electric field in a dielectric is sufficiently large, it begins to pull electrons completely out of the molecules, and the dielectric becomes conducting. Dielectric breakdown is said to have occurred when a dielectric becomes conducting. Dielectric breakdown occurs in all kinds of dielectric materials (gases, liquids, or solids) and depends on the nature of the material, temperature, humidity, and the amount of time that the field is applied. The minimum value of the electric field at which dielectric breakdown occurs is called the dielectric strength of the dielectric material.

The dielectric strength is the maximum electric field that a dielectric can tolerate or withstand without breakdown.

5.0.1 Linear, Isotropic, And Homogeneous Dielectrics

. A material is said to be linear if D varies linearly with E and nonlinear otherwise. Materials for which ϵ (or σ) does not vary in the region being considered and is therefore the same at all points (i.e., independent of x, y, z) are said to be homogeneous. They are said to be inhomogeneous (or non homogeneous) when ϵ is dependent of the space coordinates. The atmosphere is a typical example of an inhomogeneous medium; its permittivity varies with altitude. Materials for which D and E are in the same direction are said to be isotropic. That is, isotropic dielectrics are those which have the same properties in all directions. For anisotropic (or non isotropic) materials, D, E, and P are not parallel; ϵ or χ_e has nine components that are collectively referred to as a tensor. Crystalline materials and magnetized plasma are anisotropic.

A dielectric material (in which $D = \epsilon E$ applies) is linear if ϵ does not change with the applied E field. Homogeneous if ϵ does not change from point lo point, and isotropic if ϵ does not change with direction.

5.0.2 Continuity Equation And Relaxation Time

Due to the principle of charge conservation, the time rate of decrease of charge within a given volume must be equal to the net outward current flow through the closed surface of the volume. Thus current I_{out} coming out of the closed surface is

$$I_{out} = \oint J \bullet ds = -\frac{dQ_{in}}{dt} \tag{5.8}$$

where Q in is the total charge enclosed by the closed surface. Invoking divergence theorem

$$\oint_s J \bullet ds = \int_v \nabla \bullet J dv \tag{5.9}$$

but

$$-\frac{dQ_{in}}{dt} = -\frac{d}{dt}\int_v \rho_v dv = -\int_v \frac{\partial \rho_v}{\partial t} dv \tag{5.10}$$

substituting

$$\int_v \nabla \bullet J dv = -\int_v \frac{\partial \rho_v}{\partial t} dv \tag{5.11}$$

or

$$\nabla \bullet J = -\frac{\partial \rho}{\partial t} \tag{5.12}$$

which is called the continuity of current equation. It must be kept in mind that the equation is derived from the principle of conservation of charge and essentially states that there can be no accumulation of charge at any point. For steady currents , $\frac{\partial \rho_v}{\partial t} = 0$ and hence $\nabla \bullet J = 0$ showing that the total charge leaving a volume is the same as the total charge entering it.

As

$$J = \sigma E \tag{5.13}$$

and Gauss law is

$$\nabla \bullet E = \frac{\rho_v}{\epsilon} \tag{5.14}$$

Substituting the known relations

$$\nabla \bullet \sigma E = \frac{\sigma \rho_v}{\epsilon} = -\frac{\partial \rho_v}{\partial t} \tag{5.15}$$

or

$$\frac{\partial \rho_v}{\partial t} + \frac{\sigma}{\epsilon}\rho_v = 0 \tag{5.16}$$

this is a homogeneous linear ordinary differential equation . This can be solved by using separation of variables method

$$\begin{aligned}
\frac{\partial \rho_v}{\rho_v} &= -\frac{\sigma}{\epsilon}\partial t \\
\ln \rho_v &= -\frac{\sigma t}{\epsilon} + \ln \rho_{v0} \\
\rho_v &= \rho_{v0} e^{-\frac{t}{T_r}}
\end{aligned}$$

where

$$T_r = \frac{\epsilon}{\sigma} \tag{5.17}$$

ρ_{v0} is the initial charge density (i.e., ρ_v at $t = 0$). The equation shows that as a result of introducing charge at some interior point of the material there is a decay of volume charge density ρ_v . Associated with the decay is charge movement from the interior point at which it was introduced to the surface of the material. The time constant T_r (in seconds) is known as the ***relaxation time*** or ***rearrangement time.***

Relaxation time is the time it takes a charge placed in the interior of a material to drop to $e^{-1} = 36.8$ ***percent of its initial value.***

It is small for good conductors and large for good dielectrics. For example , for copper $\sigma = 10^{-7}$ mhos/m and $\epsilon_r = 1$ and

$$T_r = \frac{\epsilon_r \epsilon_0}{\sigma} = 1 \times \frac{10^{-9}}{36\pi} \times \frac{1}{5.8 \times 10^7} = 1.53 \times 10^{-19}\, sec \tag{5.18}$$

For fused quartz for instance ,$\sigma = 10^{-7} mhos/sec$, $\epsilon_r = 5.0$

$$T_r = 5 \times \frac{10^{-9}}{36\pi} \times \frac{1}{10^{-17}} = 51.5 \text{ Days} \tag{5.19}$$

showing a very large relaxation time. For good dielectrics , one may consider the introduced charge to remain where placed.

5.0.3 Boundary Conditions:

If the field exists in a region consisting of two different media , the conditions that the field must satisfy at the interface separating the media are called boundary conditions. These conditions are helpful in determining the field on one side of the boundary if the field on the other side is known. The conditions will be dictated by types of materials the media is made of. We will consider the boundary conditions at an interface separating

- Dielectric(ϵ_{r1}) and dielectric (ϵ_{r2})
- Conductor and dielectric
- Conductor and free space

To determine the boundary conditions , we need to use Maxwell's equations

$$\oint E \bullet dl = 0 \tag{5.20}$$

and

$$\oint D \bullet ds = Q_{enc} \tag{5.21}$$

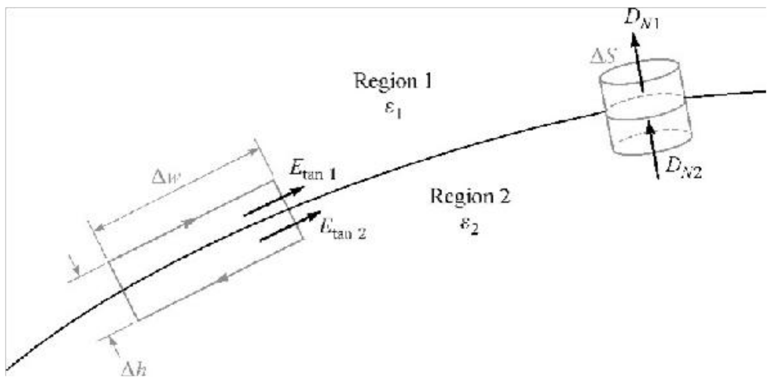
Also we need to decompose the electric field intensity E into two orthogonal components

$$E = E_t + E_n \tag{5.22}$$

where E_t and E_n are , respectively, the tangential and normal components of E to the interface of interest. A similar decomposition can be done for the electric flux density D .

5.0.3.1 Dielectric-Dielectric Boundary Conditions

Consider the E field existing in a region consisting of two different dielectrics characterized by $\epsilon_1 = \epsilon_0\epsilon_{r1}$ and $\epsilon_2 = \epsilon_0\epsilon_{r2}$ as shown in the figure.



E_1 and E_2 in media 1 and 2 , respectively , can be decomposed as

$$E_1 = E_{1t} + E_{1n}$$
$$E_2 = E_{2t} + E_{2n}$$

we apply the the equation (1) to the closed path $abcda$ in the figure assuming that the path is very small wit respect to the variation of E.We obtain

$$0 = E_{1t}\Delta w - E_{1n}\frac{\Delta h}{2} - E_{2n}\frac{\Delta h}{2} - E_{2t}\Delta w + E_{2n}\frac{\Delta h}{2} + E_{1n}\frac{\Delta h}{2} \tag{5.23}$$

where $E_t = |\mathbf{E}_t|$ and $E_n = |\mathbf{E_n}|$. As $\Delta h \to 0$, the above equation becomes

$$E_{1t} = E_{2t} \tag{5.24}$$

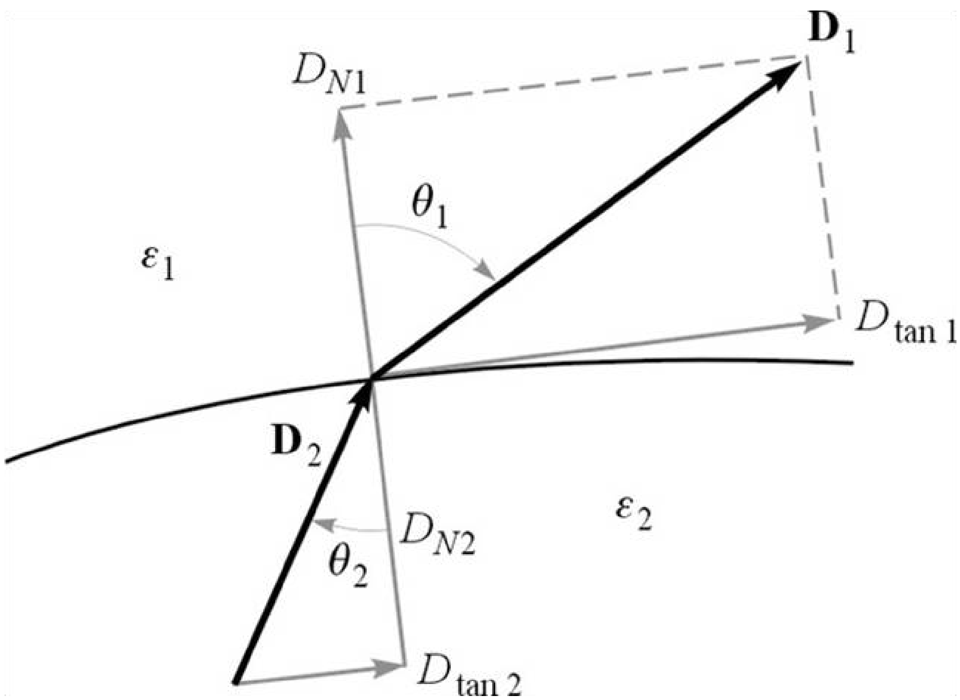
Thus the tangential component of **E**are the same on the two sides of the boundary. In other words , $\mathbf{E_t}$undergoes no change on the boundary and it is said to be continuous across the boundary . Since $\mathbf{D} = \epsilon\mathbf{E} = \mathbf{D_t} + \mathbf{D_n}$ we can write

$$\frac{D_{1t}}{\epsilon_1} = E_{1t} = E_{2t} = \frac{D_{2t}}{\epsilon_2} \tag{5.25}$$

or

$$\frac{D_{1t}}{\epsilon_1} = \frac{D_{2t}}{\epsilon_2} \tag{5.26}$$

that is , D undergoes some change across the interface. Hence D_t is said to be discontinuous across the interface.



Similarly, apply equation (2), to the pillbox (Gaussian surface). Allowing $\Delta h \to 0$ gives

$$\Delta Q = \rho_s \Delta s = D_{1n}\Delta s - D_{2n}\Delta s \tag{5.27}$$

or

$$D_{1n} - D_{2n} = \rho_s \tag{5.28}$$

where ρ_s is the free charge density placed deliberately at the boundary. If no free charge exists at the interface (ie., charges are not placed deliberately placed at the interface) , $\rho_s = 0$ and the equation becomes

$$D_{1n} = D_{2n} \tag{5.29}$$

Thus the normal component of **D** is continuous across the boundary. Since $\mathbf{D} = \epsilon \mathbf{E}$, the above equation can be written as

$$\epsilon_1 E_{1n} = \epsilon_2 E_{2n} \tag{5.30}$$

showing that the normal component of **E** is discontinuous at the boundary. The above relations are collectively called the *boundary conditions;* they must be satisfied by an field at the boundary separating two different dielectrics.

These conditions can be combined to show the change in the vectors **D** and **E** at the interface. Let $\mathbf{D_1}$ and ($\mathbf{E_1}$) make an angle θ_1 with the normal to the surface as shown in the figure above. Since the normal components of **D** are continuous

$$D_{N1} = D_1 \cos\theta_1 = D_2 \cos\theta_2 = D_{N2} \tag{5.31}$$

the ratio of tangential components is given by

$$\frac{D_{tan1}}{D_{tan2}} = \frac{D_1 \sin\theta_1}{D_2 \sin\theta_2} = \frac{\epsilon_1}{\epsilon_2} \tag{5.32}$$

or

$$\epsilon_2 D_1 \sin\theta_1 = \epsilon_2 D_2 \sin\theta_2 \tag{5.33}$$

Combining the equations

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\epsilon_1}{\epsilon_2} \tag{5.34}$$

In the above figure , we have assumed that $\epsilon_1 > \epsilon_2$, and therefore $\theta_1 > \theta_2$.

The direction of $\mathbf{E}$ on each side of the boundary is identical with the direction of $\mathbf{D}$, because $\mathbf{D} = \epsilon\mathbf{E}$. The magnitude of $\mathbf{D}$ in region 2 may be found as

$$D_2 = D_1\sqrt{\cos^2\theta_1 + \left(\frac{\epsilon_2}{\epsilon_1}\right)^2 \sin^2\theta_1} \tag{5.35}$$

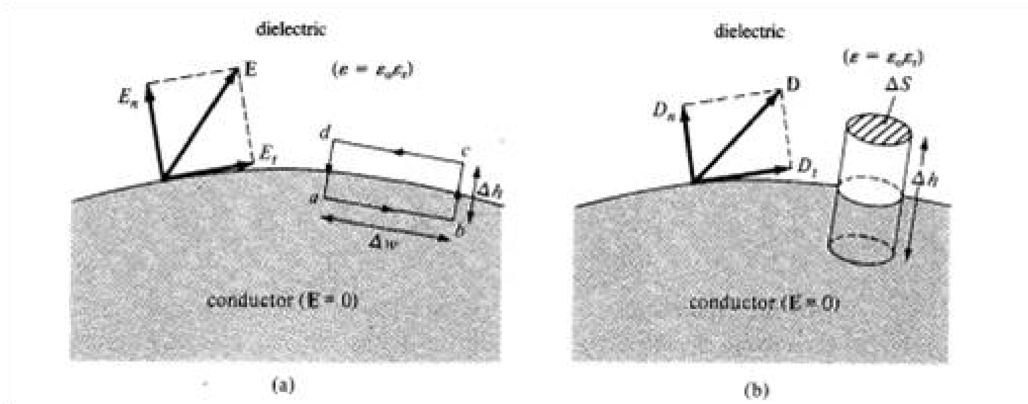
and the magnitude of $\mathbf{E_2}$ is

$$E_2 = E_1\sqrt{\sin^2\theta_1 + \left(\frac{\epsilon_1}{\epsilon_2}\right)^2 \cos^2\theta_1} \tag{5.36}$$

An inspection of these equations shows that D is larger in the region of larger permittivity (unless $\theta_1 = \theta_2 = 0$, where the magnitude is unchanged) and that E is larger in the region of smaller permittivity (unless $\theta_1 = \theta_2 = 90^0$, where its magnitude is unchanged). These boundary conditions, or the magnitude and direction relations derived from them, allow us to find quickly the field on one side of the boundary if we know the field on the other side.

5.0.3.2 Conductor - Dielectric Boundary:

This is the case shown in figure below.



The conductor is assumed to be perfect. Although such a conductor is not practically realizable , we may regard conductors such as copper and silver as though they were perfect conductors. To determine the boundary conditions for a conductor-dielectric interface , we follow the same procedure used for dielectric -dielectric interface except that we incorporate the fact that $E = 0$ inside the conductor. For the closed path *abcda*

$$0 = 0.\Delta w + 0.\frac{\Delta h}{2} + E_n.\frac{\Delta h}{2} - E_t.\Delta w - E_n.\frac{\Delta h}{2} - 0.\frac{\Delta h}{2} \quad (5.37)$$

as $\Delta h \to 0$

$$E_t = 0 \quad (5.38)$$

Similarly, for the pillbox letting $\Delta h \to 0$, we get

$$\Delta Q = D_n.\Delta s - 0.\Delta s \quad (5.39)$$

because $\mathbf{D} = \epsilon\mathbf{E} = \mathbf{0}$ inside the conductor. Then

$$D_n = \frac{\Delta Q}{\Delta s} = \rho_s \quad (5.40)$$

or

$$D_n = \rho_s \quad (5.41)$$

Thus under static conditions , the following conclusions can be made about a perfect conductor

1. No electric field may exist within a conductor : that is

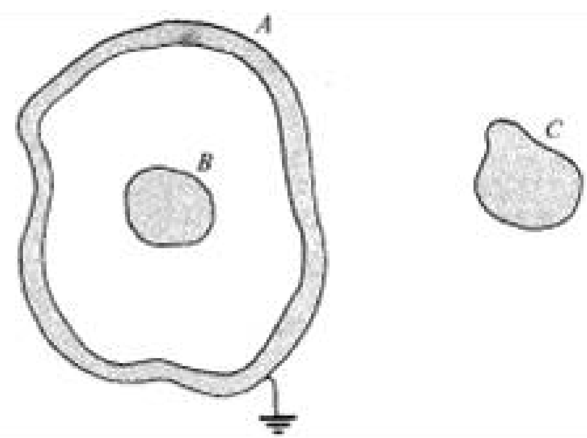
$$\rho_s = 0 \quad E = 0 \tag{5.42}$$

2. Since $\mathbf{E} = -\nabla \mathrm{V} = 0$, there can be no potential difference between any two points in the conductor, that is a conductor is an equipotential surface.

3. The electric field $\mathbf{E}$ can be external to the conductor and normal to the surface: that is

$$D_t = \epsilon_0 \epsilon_r E_t = 0 \quad D_n = \epsilon_0 \epsilon_r E_n = \rho_s \tag{5.43}$$

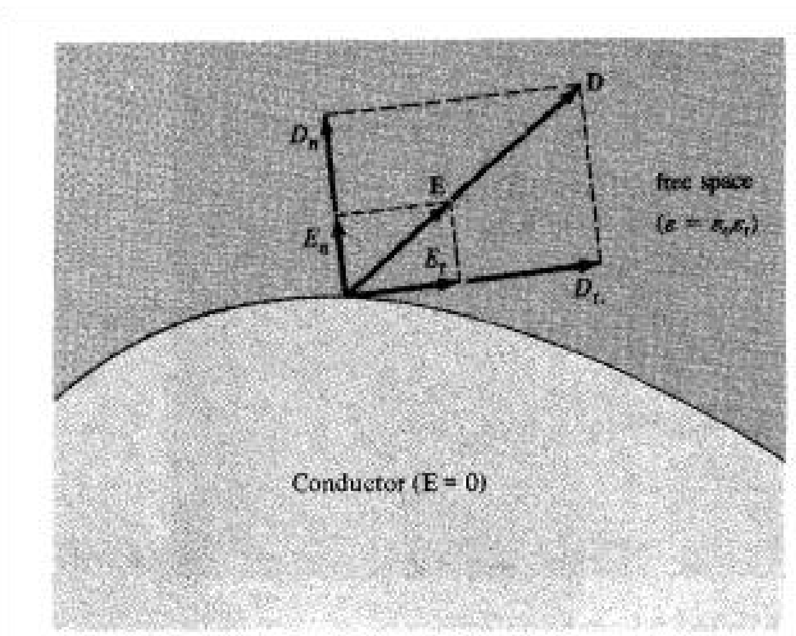
An important application of the fact that $\mathbf{E} = \mathbf{0}$ inside a conductor is in electrostatic screening or shielding.



If conductor A kept at zero potential surrounds conductor B as shown in the figure above , B is said to be electrostatically screened by A from other electric systems such as C outside A . Similarly, Conductor C outside A is screened by A from b . The conductor A acts like a screen or shield and the electrical conditions inside and outside the screen are completely independent of each other.

5.0.3.3 Conductor Free space Boundary Conditions

This is a special case of the conductor-dielectric conditions and is shown in the figure below.



The boundary conditions at the interface between a conductor and free space can be obtained by replacing $\epsilon_r = 1$. We expect the electric field **E** to be external to the conductor and normal to its surface. Thus the boundary conditions are

$$D_t = \epsilon_0 E_t = 0 \quad D_n = \epsilon_0 E_n = \rho_s \tag{5.44}$$

it should be noted that the above equation implies that **E** field must approach a conducting surface normally.

5.1 Capacitance

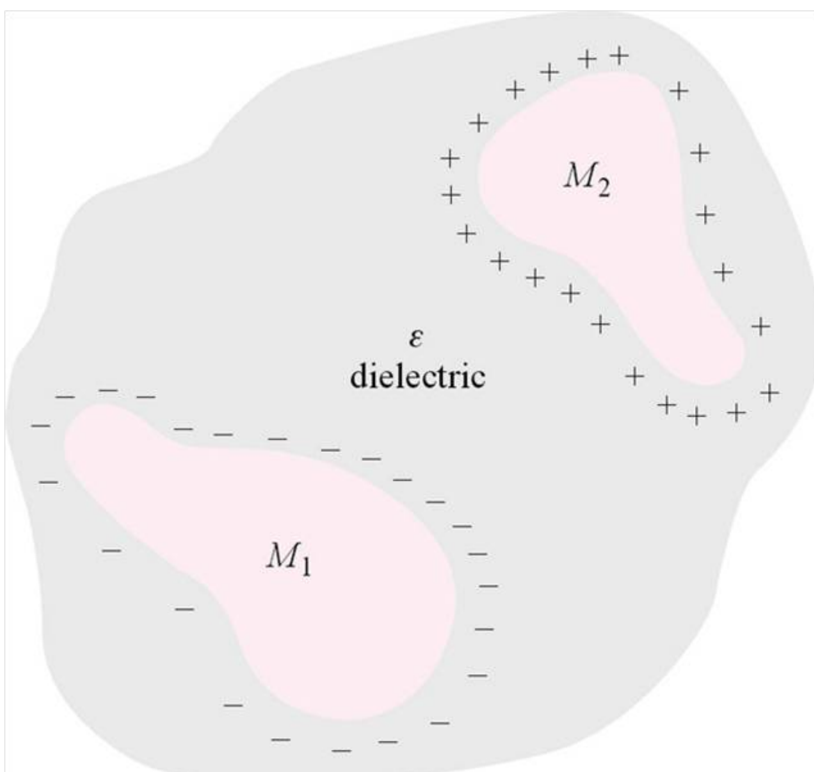
A capacitor is an electrical device composed of two conductors which are separated by a dielectric medium and which can store equal and opposite charges, independently of whether other conductors in the system are charged or not.

The capacitance between two conducting bodies is defined as

$$C = \frac{Q}{V} \tag{5.45}$$

Q is charge in Coulombs and V is the potential difference between conducting bodies. When the capacitance of a single conductor is referred to, it is tacitly assumed that the other conductor is a spherical shell of infinitely large radius.

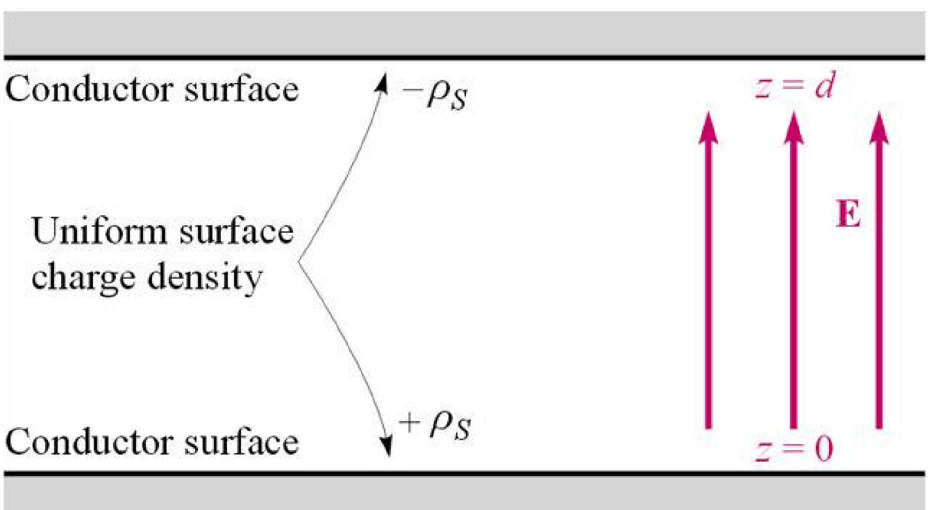
Consider conductors 1 and 2 of arbitrary shape, as shown below



Work is done on moving a charge from one conductor to the other. consequently a potential difference is established between them. Conversely a P.D of V volts is applied between the conductors 1 and 2 , charges Q and $-q$ will be built up on the conductors. there is a definite relationship between the charge and the Potential difference and the ratio between the two is constant, determined from the geometrical configuration of any particular system of conductors.

If a charge of 1 Coulomb is associated with a voltage of $1\,V$, the capacitance between the conductors is 1 Farad . 1 Farad is a very large quantity so capacitance is normally given in terms of micro, or pico Farads.

5.1.1 Parallel plate capacitor



Assume that the charge density on the plates is equal to $\rho_s\,C/m^2$. The dielectric has $\epsilon = \epsilon_r\epsilon_0$, then

$$D = \rho_s = \frac{Q}{A}$$
$$E = \frac{D}{\epsilon} = \frac{\rho_s}{\epsilon_r \epsilon_0}$$

Potential difference between the plates is given by the integral of E over the separation of the plates d

$$V = Ed = \frac{\rho_s}{\epsilon_r \epsilon_0} \tag{5.46}$$

$$C = \frac{Q}{V} = \frac{\rho_s A}{\frac{\rho_s d}{\epsilon_r \epsilon_0}} = \frac{\epsilon_r \epsilon_0}{d} \tag{5.47}$$

5.1.2 Spherical Capacitor

$$E = \frac{Q}{4\pi\epsilon_0 r^2} \quad a < r < b$$
$$V_{ba} = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right]$$
$$C = \frac{Q}{V_{ba}} = \frac{Q}{\frac{Q}{4\pi\epsilon_0\left[\frac{1}{a}-\frac{1}{b}\right]}} = \frac{4\pi\epsilon_0 ab}{(b-a)}$$

as $b \longrightarrow \infty$, C for an isolated sphere is $4\pi\epsilon_0 a$ Farads .

5.1.3 ENERGY STORED IN AN ELECTROSTATIC FIELD:

The amount of work necessary to assemble a group of point charges equals the total energy (W_e) stored in the resulting electric field.

Example (3 point charges): Given a system of 3 point charges, we can determine the total energy stored in the electric field of these point charges by determining the work performed to assemble the charge distribution. We first define V_{mn} as the absolute potential at P_m due to point charge Q_n.

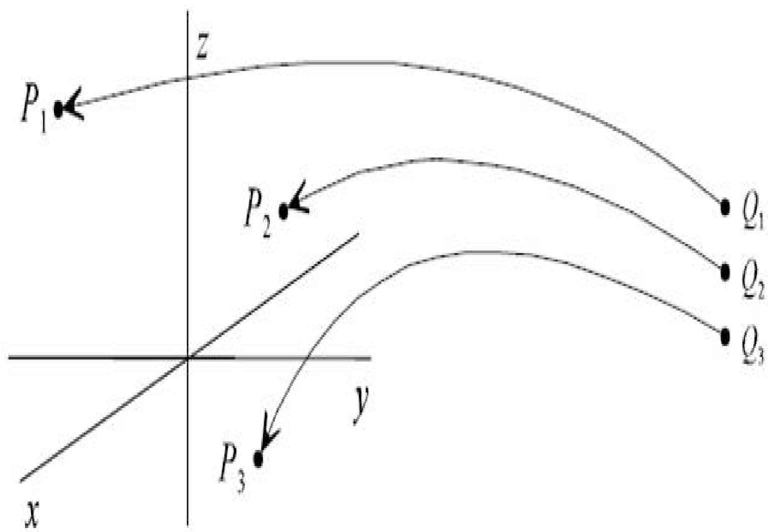


Figure 5.1: Energy to move point charges

1. Bring Q_1 to P_1(no energy required).
2. Bring Q_2 to P_2 (work $= Q_2V_{21}$).
3. Bring Q_3 to P_3 (work $= Q_3V_{31} + Q_3V_{32}$)

The total work done $W_e = 0 + Q_2V_{21} + Q_3V_{31} + Q_3V_{32}$

If we reverse the order in which the charges are assembled, the total energy required is the same as before.

1. Bring Q_3 to P_3 (No energy required)

2. Bring Q_2 to P_2 (work=Q_2V_{23})

3. Bring $Q_1 to P_1$ (work done = $Q_1V_{12} + Q_1V_{13}$)

Total work done $W_e = 0 + Q_2V_{23} + Q_1V_{12} + Q_1V_{13}$

Adding the above two equations

$$2W_e = Q_1V_{12} + Q_1V_{13} + Q_2V_{21} + Q_2V_{23} + Q_3V_{31} + Q_3V_{32} \quad (5.48)$$

$$W_e = \frac{1}{2}\left[(Q_1(V_{12} + V_{13}) + Q_2(V_{21} + V_{23}) + Q_3(V_{31} + V_{32})\right] = \frac{1}{2}\left[Q_1V_1 + Q_2V_2 + Q_2V_3 \quad (5.49)$$

where V_m is the total absolute potential at P_m affecting Q_m.

In general, for a system of N point charges, the total energy in the electric field is given by

$$W_e = \frac{1}{2}\sum_{k=1}^{N} Q_kV_k \quad (5.50)$$

For line, surface or volume charge distributions, the discrete sum total energy formula above becomes a continuous sum (integral) over the respective charge distribution. The point charge term is replaced by the appropriate differential element of charge for a line, surface or volume distribution: $\rho_L dL$, $\rho_s ds$ or $\rho_v dv$. The overall potential acting on the point charge Q_k due to the other point charges (V_k) is replaced by the overall potential (v) acting on the differential element of charge due to the rest of the charge distribution. The total energy expressions becomes

$$W_e = \frac{1}{2}\int_L \rho_L dL \qquad (Line\ Charge) \quad (5.51)$$

$$W_e = \frac{1}{2}\int_s \rho_s ds \qquad (Surface\ Charge) \tag{5.52}$$

$$W_e = \frac{1}{2}\int_v \rho_v dv \qquad (Volume\ Charge) \tag{5.53}$$

If a volume charge distribution ρ_v of finite dimension is enclosed by a spherical surface S_0 of radius r_0, the total energy associated with the charge is given by

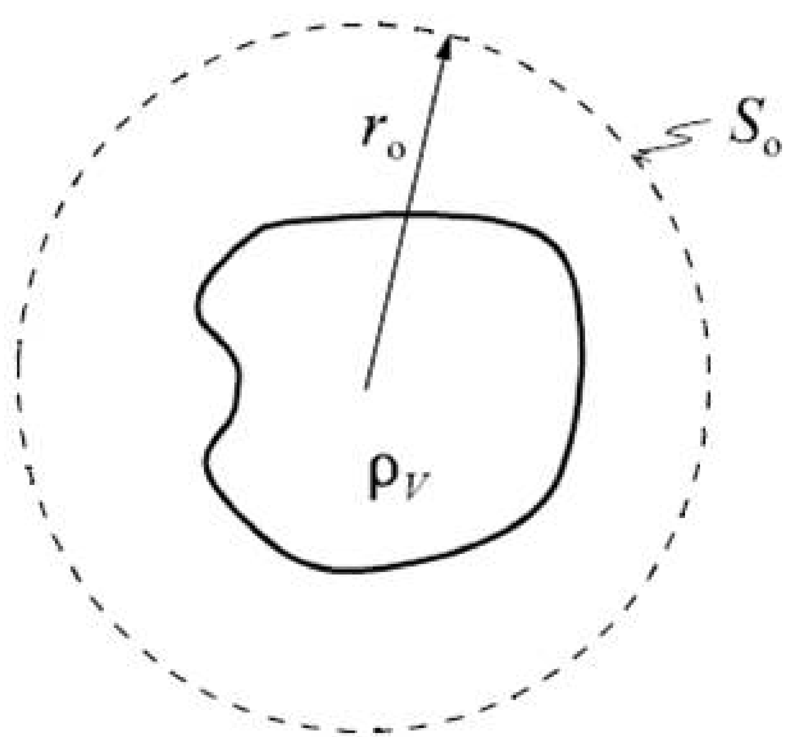


Figure 5.2: Distribution of volume charge

$$W_e = \lim_{r_0 \to \infty} \left[\frac{1}{2} \int_v \rho_v V dv \right] = \lim_{r_0 \to \infty} \left[\frac{1}{2} \int (\nabla \bullet D) V dv \right] \quad (5.54)$$

Using the following vector identity,

$$(\nabla \bullet D)V = \nabla \bullet (VD) - D \bullet \nabla V \quad (5.55)$$

the expression for the total energy can be written as

$$W_e = \lim_{r_0 \to \infty} \left[\frac{1}{2} \int_v [\nabla \bullet (VD)] dv \right] - \lim_{r_0 \to \infty} \left[\frac{1}{2} \int_v (D \bullet \nabla V) dv \right] \quad (5.56)$$

If we apply the divergence theorem to the first integral, we find

$$W_e = \lim_{r_0 \to \infty} \left[\frac{1}{2} \oiint VD \bullet ds \right] - \lim_{r_0 \to \infty} \left[\frac{1}{2} \int_v (D \bullet \nabla V) dv \right] \quad (5.57)$$

For each equivalent point charge ($\rho_v dv$) that makes up the volume charge distribution, the potential contribution on S_0 varies as r^{-1} and electric flux density (and electric field) contribution varies as r^{-2}. Thus, the product of the potential and electric flux density on the surface So varies as r^{-3}. Since the integration over the surface provides a multiplication factor of only r^2, the surface integral in the energy equation goes to zero on the surface S_0 of infinite radius. This yields where the integration is applied over all space. The divergence term in the integrand can be written in terms of the electric field as

$$E = -\nabla V \quad (5.58)$$

such that the total energy (J) in the electric field is

$$W_e = \frac{1}{2}\iiint_v D \bullet E dv = \frac{1}{2}\iiint_v \epsilon_0 (E \bullet E) dv = \frac{1}{2}\iiint_v \epsilon_0 E^2 dv \tag{5.59}$$

This can also be expressed as

$$\frac{dW_e}{dv} = \frac{1}{2}\epsilon_0 E^2 \tag{5.60}$$

$\frac{dW_e}{dv}$ is called the energy density and is given in J/m^3.

5.2 Current and Current density:

Electrical charges in motion constitute current. The unit of current is Ampere and is defined as the rate of movement of charge passing a given reference point (or passing a given reference plane) of one coulomb/sec. Current is denoted by I

$$I = \frac{dQ}{dt} \tag{5.61}$$

Current is thus defined by the motion of the positive charges, even though conduction in metals takes place through the motion of electrons.

In field theory events occurring at a point, rather than within a small region are of interest, and the concept of current density, measured in Amperes/sq.m will be more useful. Current density is a vector represented by J .

The increment of current ΔI crossing an incremental surface area Δs normal to the current density is $\Delta I = J_N \triangle s$ and in the case where the current density is not perpendicular to the surface,

$$\Delta I = J \bullet \Delta s \tag{5.62}$$

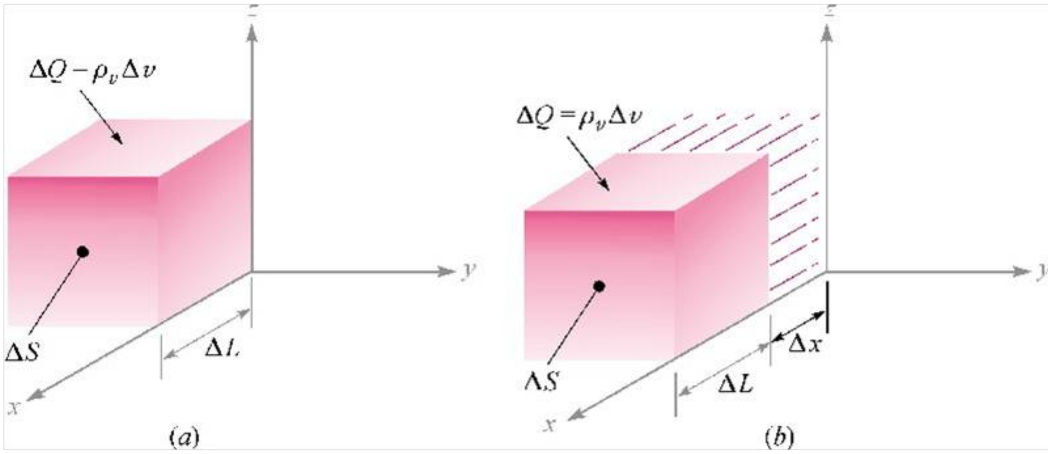


Figure 5.3:

total current is obtained by integrating

$$I = \int_s J \bullet ds \tag{5.63}$$

current density is related to the velocity of volume charge density at a point. Consider the element of charge

$$\Delta Q = \rho_v \Delta v = \rho_v \Delta s \Delta L \tag{5.64}$$

let us assume that the charge element is oriented with its edges parallel to the coordinate axes, and that it possesses only an x component of velocity. In the time interval Δt , the element of charge has moved a distance Δx , as indicated in the figure . We have therefore moved a charge $\Delta Q = \rho_v \Delta s \Delta x$ through a reference plane perpendicular to the direction of motion in a time increment Δt , and the resultant current is

$$\Delta I = \frac{\Delta Q}{\Delta t} = \rho_v \Delta s \frac{\Delta x}{\Delta t} \tag{5.65}$$

in the limit as $\Delta t \longrightarrow 0$ we have

$$\Delta I = \rho_v \Delta s v_x \tag{5.66}$$

where $v_x$is the velocity in the $x-$ direction. In terms of the current density

$$J_x = \rho_v v_x \tag{5.67}$$

and in general

$$J - \rho_v v \tag{5.68}$$

The above result shows very clearly that charge in motion constitutes a current . This type of current is called convection current density. The convection current density is related linearly to charge density as well as to velocity.

5.2.1 Continuity Of current:

The principle of charge conservation states that charge can neither be created nor destroyed , although equal amounts of positive and negative charge may be simultaneously created, obtained by separation, destroyed or lost by recombination.

The continuity equation follows from this principle when we consider any region bounded by a closed surface. The current through the closed surface is

$$I = \oint J \bullet ds \tag{5.69}$$

and this outward flow of positive charge must be balanced by a decrease of positive charge (or perhaps an increase of negative charge) within the closed surface. if the charge inside the closed surface is denoted by Q , then the rate of decrease is $-\frac{dQ_i}{dt}$ and the principle of charge conservation requires that

$$I = \oint J \bullet ds = -\frac{dQ}{dt} \tag{5.70}$$

The above equation is the integral form of the continuity equation. The point form is obtained from the above by using the divergence theorem.

$$\oint J \bullet ds = \int_v (\nabla \bullet J)\, dv$$

$$-\frac{dQ}{dt} = -\frac{d}{dt}\int_v \rho_v dv = -\int_v \frac{\partial \rho_v}{\partial t} dv$$

$$\int_v (\nabla \bullet J)\, dv = -\int_v \frac{\partial \rho_v}{\partial t} dv$$

This is true for any volume dv . This is possible only if the integrands are equal. so

$$\nabla \bullet J = -\frac{\partial \rho}{\partial t} \tag{5.71}$$

From the physical interpretation of divergence, the above equation indicates that the current or charge per second diverging from a small volume per unit volume is equal to the time rate of decrease of charge per unit volume at every point.

5.2.2 Ohm's Law: Point Form

Consider a conductor. the valance electrons , or conduction or free electrons , move under the influence of an electric field . With a field E , an electron having a charge $Q = -e$ will experience a force ,

$$F = -eE \tag{5.72}$$

In free space the electron will accelerate and continually increase its velocity (and energy) . In the crystalline material the progress of the electron is impeded by continual collisions with the thermally excited crystalline lattice structure, and a constant average velocity is soon attained . This velocity V_d is termed as the drift velocity and is linearly related to the electric field intensity by the mobility of the electron in the given material. Mobility is denoted by μ

$$V_d = -\mu_e E \tag{5.73}$$

The electron velocity is in a direction opposite to the direction of E . μ_e has the dimensions of square meter/Volt-second. Typical values are

Aluminum	0.0012
Copper	0.0032
silver	0.0056

For good conductors a drift velocity of a few inches per second is sufficient to produce a noticeable temperature rise and can cause the wire to melt if the heat can not be quickly removed by thermal conduction.

Substituting in J

$$J = -\mu_e \rho_e E \tag{5.74}$$

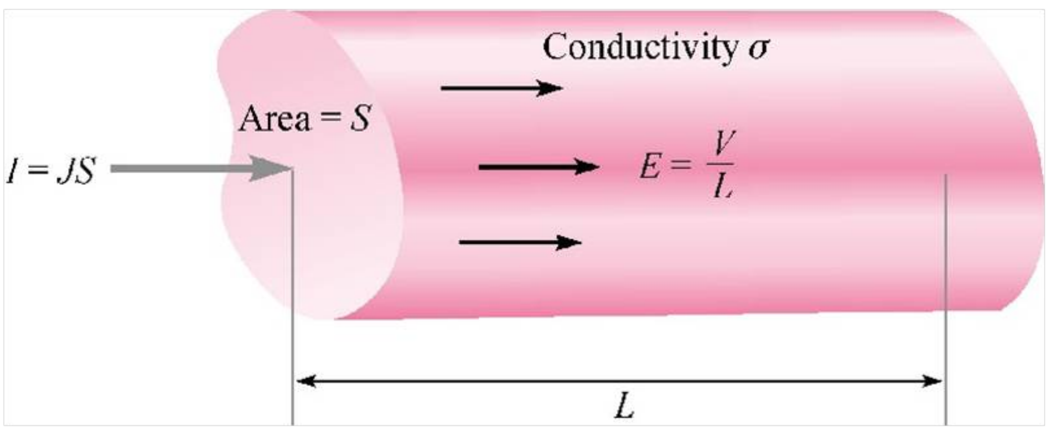
The relationship between J and E for metallic conductors , is also specified in terms of conductivity as

$$J = \sigma E \tag{5.75}$$

where σ is in mho/m. The above relation is called the point form of Ohm's law. The conductivity $\sigma = -\mu_e \rho$. The values of conductivity for Aluminum, copper, and silver are

Aluminum	3.82×10^7
Copper	5.8×10^7
Silver	6.17×10^7

5.2.3 General Expression for Resistance



$$
\begin{aligned}
I &= \int_s J \bullet ds \\
V_{ab} &= -\int_b^a E \bullet dL = -EL_{ab} \\
V &= EL_{ab} = EL \\
J &= \frac{I}{s} = \sigma E = \sigma \frac{V}{L} \\
I &= \frac{\sigma s V}{L} \\
\frac{V}{I} &= \frac{L}{\sigma s}
\end{aligned}
$$

When the field is nonuniform, the resistance is in general given by

$$
R = \frac{V_{ab}}{I} = \frac{-\int_b^a E \bullet dl}{\int_s \sigma E \bullet ds} \qquad (5.76)
$$

Unit-IV

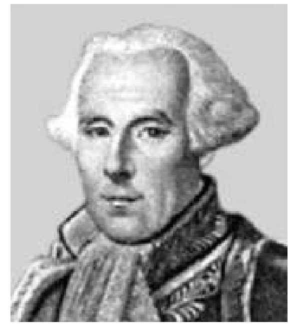
Magnetostatics:

Static magnetic fields – Biot-Savart's law – Oesterd's experiment - Magnetic field intensity (MFI) – MFI due to a straight current carrying filament – MFI due to circular, square and solenoid current – Carrying wire – Relation between magnetic flux, magnetic flux density and MFI – Maxwell's second Equation, $\nabla \bullet B = 0$.

Chapter 6

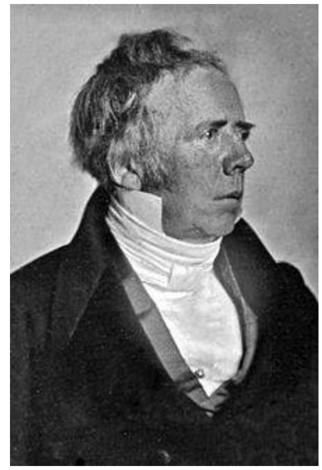
THE STEADY MAGNETIC FIELD

Jean-Baptiste Biot (21 April 1774 – 3 February 1862) was a French physicist, astronomer, and mathematician who established the reality of meteorites, made an early balloon flight, and studied the polarization of light.Jean-Baptiste Biot was born in Paris, France on 21 April 1774 and died in Paris on 3 February 1862. Biot served in the artillery before he was appointed professor of mathematics at Beauvais in 1797. He later went on to become a professor of physics at the Collège de France around 1800, and three years later was elected as a member of the Academy of Sciences.



Félix Savart (30 June 1791 - 16 March 1841) was the son of Gérard Savart, an engineer at the military school of Metz. His brother, Nicolas, student at École Polytechnique and officer in the engineering corps, did work on vibration. At the military hospital at Metz, Savart studied medicine and later he went on to continue his studies at the University of Strasbourg, where he received his medical degree in 1816 [1]. Savart became a professor at Collège de France in 1836 and was the co-originator of the Biot-Savart Law, along with Jean-Baptiste Biot. Together, they worked on the theory of magnetism and electrical currents. Their law was developed about 1820. The Biot-Savart Law relates magnetic fields to the currents which are their sources. Félix Savart also studied acoustics. He developed the Savart wheel which produces sound at specific graduated frequencies using rotating disks.

Hans Christian Ørsted (Danish pronunciation: [hans kʰʁæsd̥jan ˈœɐ̯sd̥ɛð]; often rendered Oersted in English; 14 August 1777 – 9 March 1851) was a Danish physicist and chemist who discovered that electric currents create magnetic fields, an important aspect of electromagnetism. He shaped post-Kantian philosophy and advances in science throughout the late 19th century.[1]



In 1824, Ørsted founded Selskabet for Naturlærens Udbredelse (SNU), a society to disseminate knowledge of the natural sciences. He was also the founder of predecessor organizations which eventually became the Danish Meteorological Institute and the Danish Patent and Trademark Office. Ørsted was the first modern thinker to explicitly describe and name the thought experiment.

A leader of the so-called Danish Golden Age, Ørsted was a close friend of Hans Christian Andersen and the brother of politician and jurist Anders Sandøe Ørsted, who eventually served as Danish prime minister (1853–54).

The oersted (Oe), the cgs unit of magnetic H-field strength, is named after him.

6.0.1 INTRODUCTION:

We will begin our study of the magnetic field with the definition of the magnetic field itself and show how it arises from a current distribution. The effect of this field on other currents, will also be discussed.

The relation of the steady magnetic field to its source is more complicated than is the relation of the electrostatic field to its source. It is necessary to accept several laws temporarily on faith alone. The proof of the laws does exist and can be covered at an advanced level.

6.0.2 BIOT-SAVART LAW:

The source of steady magnetic field may be a permanent magnet, an electric field changing linearly with time , or a direct current. We shall largely ignore the permanent magnet and save the time-varying electric field for a later discussion. The present discussion will be concerned about the magnetic field produced by a differential dc element in free space.

We may think of this differential current element as a vanishingly small section of a current carrying filamentary conductor, where a filamentary conductor is the limiting case of a cylindrical conductor of circular cross section as the radius approaches zero. We assume a current I flowing in a differential vector length of the filament dL. The Biot-Savart law then states that at any point P the magnitude of the magnetic field intensity produced by the differential element is proportional to the product of the current , the magnitude of the differential length, and the sine of the angle lying between the filament and a line connecting the filament to the point P where the field is desired. The magnitude of the magnetic field intensity is inversely proportional to the square of the distance from the differential element to the point P . The direction of the magnetic field intensity is normal to the plane containing the differential filament and the line drawn from the filament to the point P . Of the two possible two normals, that one is two be chosen which is in the direction of progress of a right handed screw turned from dL through the smaller angle to the line from the filament P . The proportionality constant is $\frac{1}{4\pi}$.

This can be written concisely using vector notation as

$$dH = \frac{IdL \times a_R}{4\pi R^2} = \frac{IdL \times R}{4\pi R^3} \tag{6.1}$$

The units of magnetic field intensity H are evidently amperes per meter (A/m). The geometry is illustrated in the figure below.

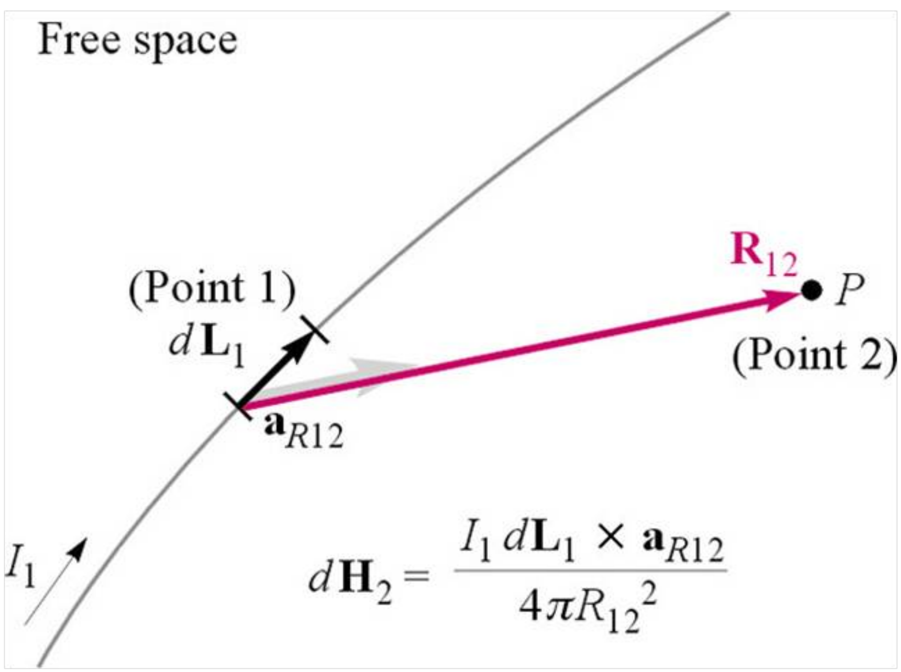


Figure 6.1: Biot-Savart Law

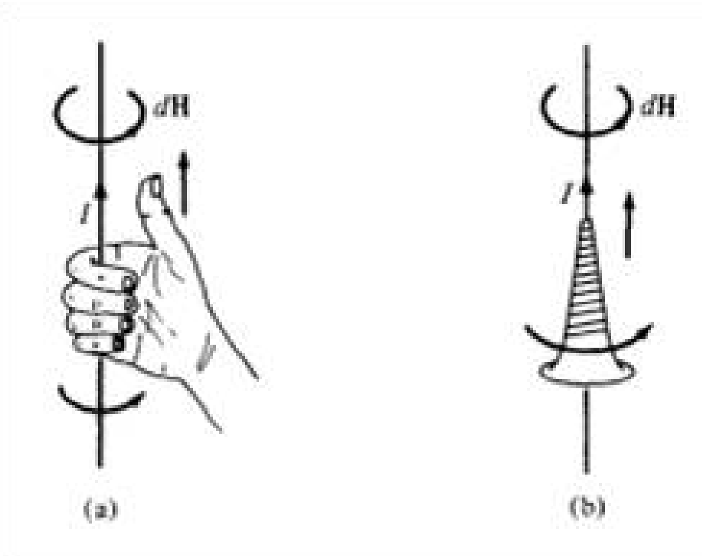
If the current element (source point) is indicated by 1 and the field point P is indicated by 2 , then

$$dH_2 = \frac{I_1 dL_1 \times a_{R12}}{4\pi R_{12}^2} \tag{6.2}$$

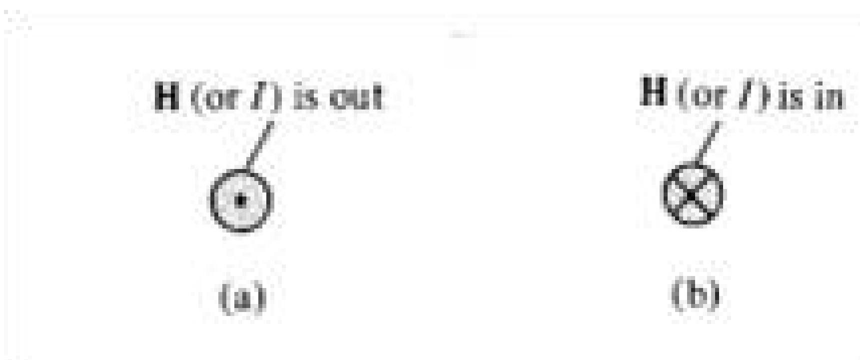
It is impossible to check Biot-Savart law in the above form because the differential current element cannot be isolated and it is an idealization. Only the integral form of the law can be verified experimentally.

$$H = \oint \frac{IdL \times a_R}{4\pi R^2} \tag{6.3}$$

The direction of **H** can be determined by the right hand rule with the right hand thumb pointing in the direction of the current , then the right hand fingers encircling the wire show the direction of **H** as shown in the figure. Alternatively , we can use the right hand screw ruleto getermine the direction of **H**. With the screw placed along the wire and pointed in the direction of the current flow, the direction of advance of the screw is the direction of **H**.



It is customary to represent the direction of the magnetic field intensity **H** (or current I) by a small circle with a dot or cross sign depending on whether **H** (or I) is out of, or into the page as illustrated in the figure.



The Biot-Savart law can also be expressed in terms of distributed sources, such as current density J and surface current density K . Surface current flows in a sheet of vanishingly small thickness , and the current density J , measured in amperes /

square meter is infinite. Surface current density , however is measured in ampere/meter width and designated by K . If the surface current density is uniform , the total current I in any width b is

$$I = Kb \tag{6.4}$$

where we have assumed that the width b is measured perpendicularly to the direction in which current is flowing. The geometry is illustrated in the figure below. For nonuniform surface current density , integration is necessary

$$I = \int K dN \tag{6.5}$$

where dN is a differential element of the path across the current flowing . so

$$IdL = Kds = Jdv \tag{6.6}$$

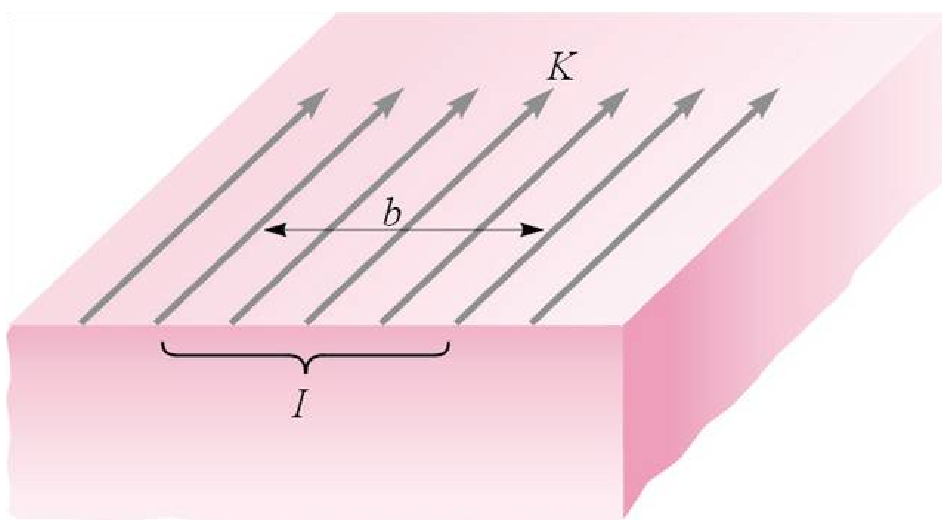


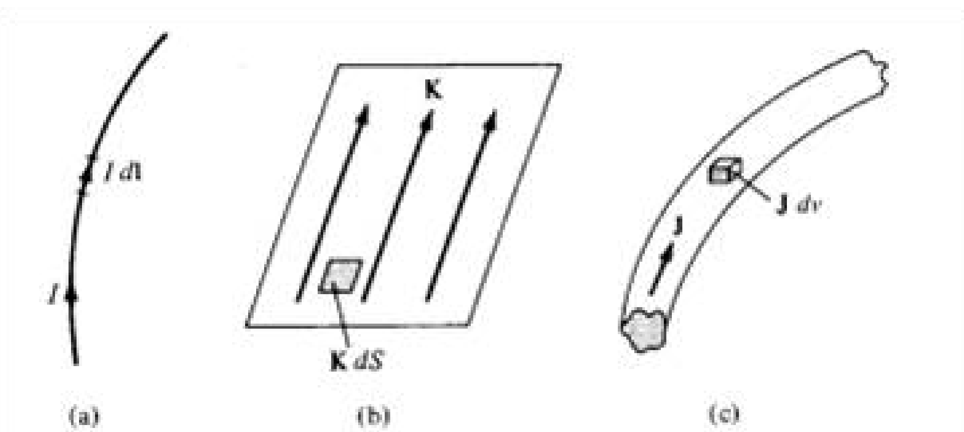
Figure 6.2: Surface current density

Biot-Savart law can be expressed in terms of current densities as

$$H = \int_s \frac{K \times a_R ds}{4\pi R^2}$$

$$H = \int_{vol} \frac{J \times a_R}{4\pi R^2}$$

Line current, surface current and volume current distributions are shown in the figure below.



6.0.3 FIELD BECAUSE OF A FINITE LINE CURRENT:

The figure below shows a finite length filamentary current

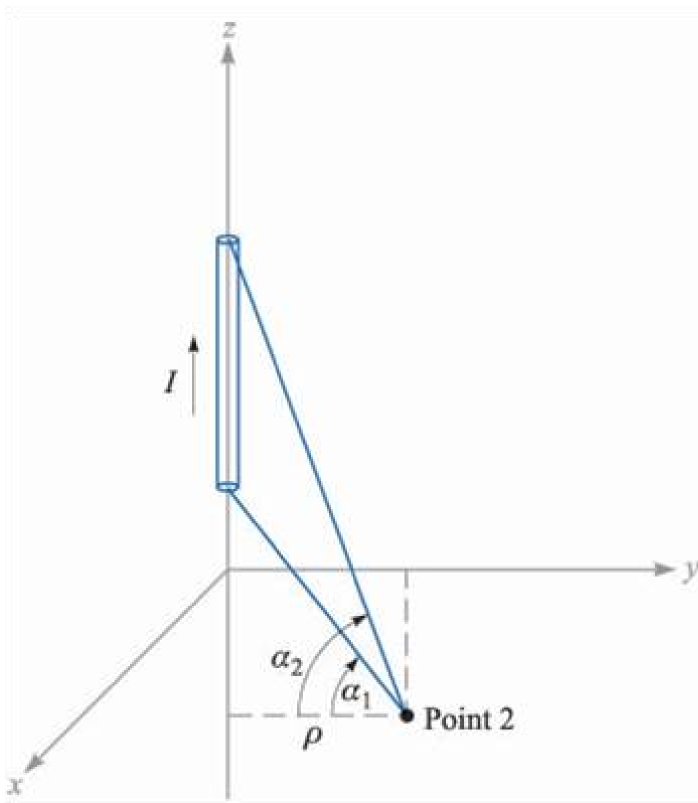


Figure 6.3: The magnetic field intensity caused by a finite length filament

Using Biot-Savart law

$$
\begin{aligned}
dH &= \frac{Idz^{'}a_z \times [\rho a_\rho + (z-z^{'})a_z]}{4\pi\left[\rho^2+(z-z')^2\right]^{\frac{3}{2}}} \\
dH &= \frac{I}{4\pi}\frac{\rho dz^{'}}{\left[\rho^2+(z-z')^2\right]^{\frac{3}{2}}}a_\phi \\
dH &= -\frac{I}{4\pi}\frac{\rho^2\sec^2\alpha d\alpha}{\rho^3\sec^3\alpha}a_\phi \\
H &= \frac{I}{4\pi\rho}\int_{\alpha_2}^{\alpha_1}(-\cos\alpha)d\alpha a_\phi \\
H &= \frac{I}{4\pi\rho}\left(\sin\alpha_2-\sin\alpha_1\right)a_\phi
\end{aligned}
$$

For an infinitely long conductor

$$\alpha_2 = 90^0\,,\ \ \alpha_1 = -90^0 \tag{6.7}$$

resulting in

$$H = \frac{I}{2\pi\rho}a_\phi \tag{6.8}$$

The magnitude of the field is not a function of ϕ or z and it varies inversely as the distance from the filament. The direction of the magnetic field intensity vector is circumferential. The streamlines are therefore circles about the filament, and the field may be mapped in cross section as shown in figure below

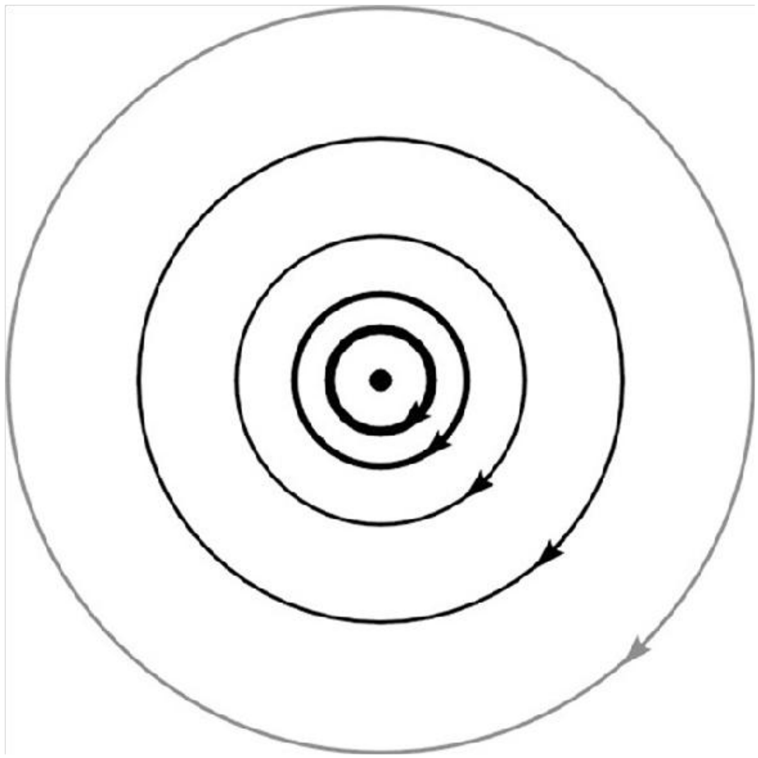


Figure 6.4: Streamlines of the magnetic field_infinitely long conductor

A comparison with the map of the electric field about an infinite line charge shows that the streamlines of the magnetic field correspond exactly to the equipotentials of the electric field, and unnamed perpendicular family of curves in the magnetic field correspond to the streamlines of the electric field.

Example;

Determine H at $P_2(0.4, 0.3, 0)$ in the field of an $8 - A$ filamentary current directed inward from infinity to the origin on the positive $x-$axis and outward to infinity along the $y-$axis . The arrangement is shown in the figure.

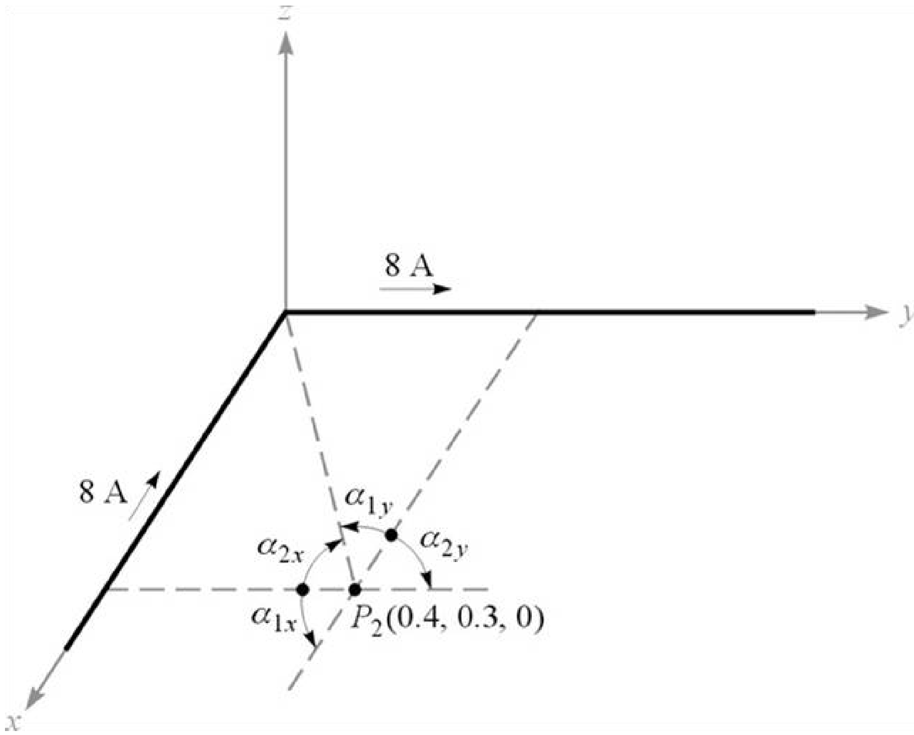


Figure 6.5:

Solution:

First consider the semi-infinite current on the $x-$axis, and identify the two angles ,$\alpha_{1x} = -90^0$ and $\alpha_{2x} = \arctan(\frac{0.4}{0.3}) = 53.1^0$. The radial distance ρ is measured from the $x-$ axis , and we have $\rho_x = 0.3$. Thus the contribution to H_2 is

$$H_{2x} = \frac{8}{4\pi(0.3)}(\sin 53.1^0 + 1)a_\phi = \frac{12}{\pi}a_\phi \qquad (6.9)$$

The unit vector a_ϕ must also be referred to the $x-$ axis . We see that this is $-a_z$. Therefore

$$H_{2x} = -\frac{12}{\pi}a_z \text{ A/m} \qquad (6.10)$$

For the current on the $y-$ axis , we have $\alpha_{1y} = -\arctan(\frac{0.3}{0.4}) = -36.9^0$ and $\rho_y = 0.4$. It follows that

$$H_{2y} = \frac{8}{4\pi(0.4)}\left(1+\sin 36.9^0\right)(-a_z) = -\frac{8}{\pi}a_z \text{ A/m} \qquad (6.11)$$

Adding these results , we have

$$H_2 = H_{2x} + H_{2y} = -\frac{20}{\pi}a_z = -6.37a_z \text{ A/m} \qquad (6.12)$$

6.0.4 MAGNETIC FIELD AT ANY POINT ON THE AXIS OF A CIRCULAR CURRENT LOOP:

Consider a circular current carrying loop shown in figure below

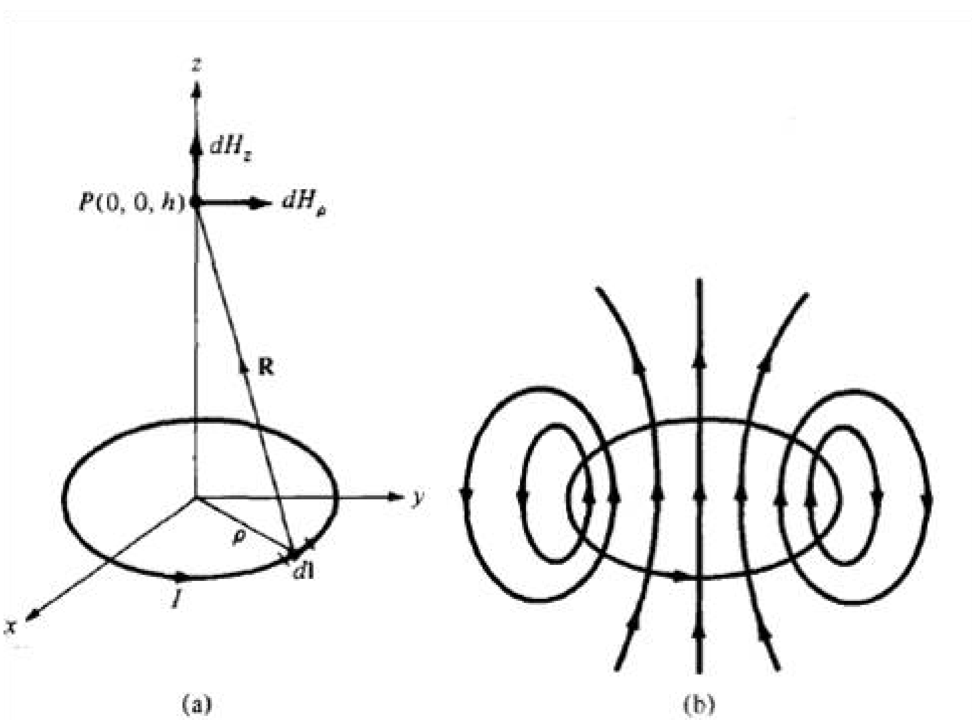


Figure 6.6: a)Circular current loop b)Flux lines due to current loop

$$
\begin{aligned}
R &= ha_z - \rho a_\rho \\
a_R &= \frac{ha_z - \rho a_\rho}{\sqrt{[h^2+\rho^2]}} \\
dH &= \frac{I\rho d\phi a_\phi \times (ha_z - \rho a_\rho)}{4\pi\left[h^2+\rho^2\right]^{\frac{3}{2}}} = \frac{I\rho h a_\rho d\phi}{4\pi\left[h^2+\rho^2\right]^{\frac{3}{2}}} + \frac{I\rho^2 a_z d\phi}{4\pi\left[h^2+\rho^2\right]^{\frac{3}{2}}} \\
H &= \int\limits_0^{2\pi}\left[\frac{I\rho h a_\rho d\phi}{4\pi\left[h^2+\rho^2\right]^{\frac{3}{2}}} + \frac{I\rho^2 a_z d\phi}{4\pi\left[z^2+\rho^2\right]^{\frac{3}{2}}}\right] = 0 + \frac{I}{4\pi}\int\limits_0^{2\pi}\frac{I\rho^2 a_z d\phi}{4\pi\left[h^2+\rho^2\right]^{\frac{3}{2}}} \\
H &= \frac{I}{2}\frac{\rho^2}{\left[h^2+\rho^2\right]^{\frac{3}{2}}}a_z
\end{aligned}
$$

Field at the center ($h = 0$) is

$$
H = \frac{I}{2\rho}a_z \tag{6.13}
$$

6.0.5 MAGNETIC FIELD AT ANY POINT ON THE AXIS OF A LONG SOLENOID:

A solenoid of length l and radius a consists of Nturns of wire and carries a current of Iamps.

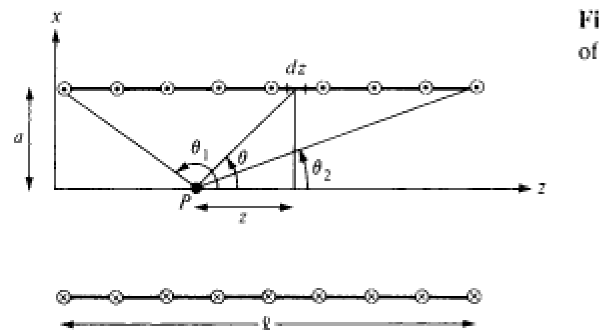


Figure 6.7: Cross section of a solenoid

Consider the cross section of the solenoid as shown in the figure above. Since the solenoid consists of circular loops , we apply the result of the circular loopto find the field. The contribution to the field **H** at P by an element of the solenoid of length dz is

$$d\mathbf{H} = \frac{Idla^2}{2\left[a^2+z^2\right]^{\frac{3}{2}}} = \frac{Ia^2ndz}{2\left[a^2+z^2\right]^{\frac{3}{2}}} \tag{6.14}$$

where $dl = ndz = \left(\frac{n}{l}\right) dz$. From the figure

$$\tan\theta = \frac{a}{z}$$

$$dz = -acosec^2\theta d\theta = -\frac{\left[z^2+a^2\right]^{\frac{3}{2}}}{a^2}\sin\theta d\theta$$

Hence

$$d\mathbf{H_z} = -\frac{nI}{2}\sin\theta d\theta \tag{6.15}$$

thus

$$\mathbf{H_z} = -\int_{\theta_1}^{\theta_2}\sin\theta d\theta \tag{6.16}$$

$$\mathbf{H} = \frac{nI}{2}\left(\cos\theta_2 - \cos\theta_1\right)\mathbf{a_z} \tag{6.17}$$

substituting $n = \frac{NI}{L}$

$$\mathbf{H} = \frac{NI}{2L}\left(\cos\theta_2 - \cos\theta_1\right)\mathbf{a_z} \tag{6.18}$$

at the center of the solenoid

$$\cos\theta_2 = \frac{\frac{L}{2}}{\left[a^2 + \frac{L^2}{4}\right]^{\frac{1}{2}}} = -\cos\theta_1 \tag{6.19}$$

and

$$\mathbf{H} = \frac{NI}{L}\cos\theta_2\mathbf{a_z} \tag{6.20}$$

If $L >> a$ or $\theta_2 = 0^0\,, \theta_1 = 180^0$

$$\mathbf{H} = \frac{NI}{L}\mathbf{a_z} \tag{6.21}$$

6.1 MAGNETIC FLUX AND MAGNETIC FLUX DENSITY:

In frre space let us define the magnetic flux density B as

$$B = \mu_0 H \tag{6.22}$$

where B is measured in Webers/square meter (Wb/m^2) or in Tesla (T).The constant μ_0 is not dimensionless and has the value

$$\mu_0 = 4\pi \times 10^{-7}\ \text{H/m} \tag{6.23}$$

This called the permeability of free space.

The magnetic flux density vector B , as the name implies, is a member of the fluxdensity family of vector fields.

If we represent magnetic flux by ϕ and define ϕ as the flux passing through any designated area

$$\phi = \int_s B \bullet ds \ \text{Wb} \tag{6.24}$$

For electrical flux the charge Q is the source of the field and the flux lines begin and terminate on positive and negative charge, respectively.

No such source has ever been discovered for the lines of magnetic flux . The magnetic flux lines are closed and do not terminate on a magnetic charge . For this reason Gauss's law for magnetic field is

$$\oint_s B \bullet ds = 0 \tag{6.25}$$

Applying divergence theorem we get

$$\nabla \bullet B = 0 \tag{6.26}$$

The above is not a proof but we have merely shown the truth.

This can also be shown starting from the Biot-Savart law. Assume that point 1 is the source point and point 2is the field point . Then Biot-Savart law in terms of volume current dnesity is given by

$$B_2 = \frac{\mu_0}{4\pi} \int_v J_1 \times \frac{R_{12}}{|R_{12}|^3} dv_1 \tag{6.27}$$

Then

$$\nabla_2 \bullet B_2 = \nabla_2 \bullet \left[\int_v J_1 \times \frac{R_{12}}{|R_{12}|^3} dv_1 \right] \tag{6.28}$$

from the vector identity

$$\nabla \bullet (x \times y) = -x \bullet \nabla \times y + y \bullet \nabla x \tag{6.29}$$

here

$$x = J_1 \ , y = \frac{R_{12}}{|R_{12}|^3} \tag{6.30}$$

So using the above relation

$$\nabla_2 \bullet B_2 = \frac{\mu_0}{4\pi} \int_v \left[-J_1 \bullet \nabla_2 \times \left(\frac{R_{12}}{|R_{12}|^3} \right) + \frac{R_{12}}{|R_{12}|^3} \bullet \nabla_2 \times (J_1) \right] dv_1 \tag{6.31}$$

$$\nabla_2 \bullet B_2 = \frac{\mu_0}{4\pi} \int_v \left[-J_1 \bullet \nabla_2 \times \nabla_2 \left(-\frac{1}{R_{12}} \right) + \frac{R_{12}}{|R_{12}|^3} \bullet \nabla_2 \times (J_1) \right] dv_1 \tag{6.32}$$

But

$$\nabla_2 \times \nabla_2 \left(-\frac{1}{R_{12}} \right) = 0 \tag{6.33}$$

as curl of gradient of any function is zero. Also

$$\nabla_2 \times (J_1) = 0 \tag{6.34}$$

as $J_!$ is a function of coordinates of point 1 and cul is taken with respect to the coordinates of point 2 . So

$$\nabla_2 \bullet B_2 = 0 \text{ or in general } \nabla \bullet B = 0 \tag{6.35}$$

The equations above are the Maxwell's equation for the steady magnetic field in integral form and differential or point form.

Collecting all the equations , both for static electric fields and steady magnetic fields we have

MAXWELL'S EQUATIONS (STATIC FIELDS)	
INTEGRAL FORM	DIFFERENTIAL OR POINT FORM
$\oint_s D \bullet ds = \int_v \rho_v dv$	$\nabla \bullet D = \rho_v$
$\oint_l E \bullet dl = 0$	$\nabla \times E = 0$
$\oint_s B \bullet ds = 0$	$\nabla \bullet B = 0$
$\oint_l H \bullet dl = \oint_s J \bullet ds$	$\nabla \times H = J$

Table 6.1: Maxwell's Equations For Static Electromagnetic Field

We will add the following equations for completeness

$$\begin{aligned} D &= \epsilon_0 E \\ B &= \mu_0 H \\ E &= -\nabla V \end{aligned}$$

Unit-V

Ampere's circuital law and its applications:

Ampere's circuital law and its applications viz. MFI due to an infinite sheet of current and a long current carrying filament – Point form of Ampere's circuital law – Maxwell's third equation, Curl (H)=Jc, Field due to a circular loop, rectangular and square loops.

Chapter 7

AMPERE'S CIRCUITAL LAW:

André-Marie Ampère (20 January 1775 – 10 June 1836) was a French physicist and mathematician who is generally regarded as one of the main founders of the science of classical electromagnetism, which he referred to as "electrodynamics". The SI unit of measurement of electric current, the ampere, is named after him.

Ampère was born on 20 January 1775 to Jean-Jacques Ampère, a prosperous businessman, and Jeanne Antoinette Desutières-Sarcey Ampère during the height of the French Enlightenment. He spent his childhood and adolescence at the family property at Poleymieux-au-Mont-d'Or near Lyon.[1] Jean-Jacques Ampère, a successful merchant, was an admirer of the philosophy of Jean-Jacques Rousseau, whose theories of education (as outlined in his treatise Émile) were the basis of Ampère's education. Rousseau believed that young boys should avoid formal schooling and pursue instead an "education direct from nature." Ampère's father actualized this ideal by allowing his son to educate himself within the walls of his well-stocked library. French Enlightenment masterpieces such as Georges-Louis Leclerc, comte de Buffon's Histoire naturelle, générale et particulière (begun in 1749) and Denis Diderot and Jean le Rond d'Alembert's Encyclopédie (volumes added between 1751 and 1772) thus became Ampère's schoolmasters. The young

Ampere's circuital law states that the line integral of H around any closed path is exactly equal to the direct current enclosed by that path.

$$\oint H \bullet dl = I \tag{7.1}$$

We define positive current as flowing in the direction of advance of a right-handed screw turned in the direction in which the closed path is traversed.

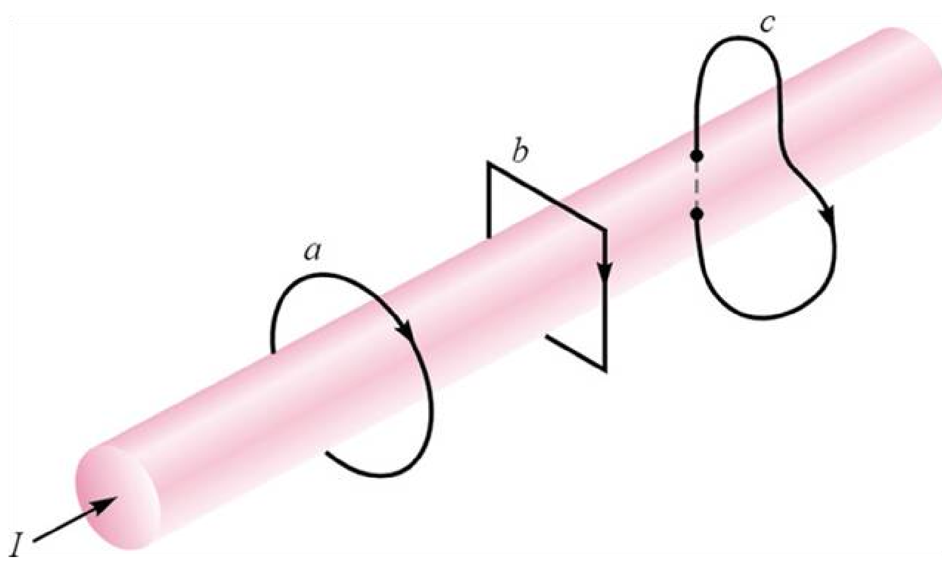


Figure 7.1: Ampere's Circuital Law

The figure shows a circular wire carrying a direct current I , the line integral of H about the closed paths lettered a and b results in an answer of I ; the integral of the closed path c which passes through the conductor gives an answer less than I and is exactly that portion of the total current which is enclosed by the path c . Although the paths a and b give the same answer , the integrands are , of course, different. The line integral directs us to multiply the component of H in the direction of the path by a small increment of path length at one point of the path, move along the path to the next incremental length, and repeat the process , continuing until the path is completely traversed. Since H will in general vary from point to point , and since paths a and b are not alike , the contributions to the integral made by, say , each millimeter of path length are quite different. Only the final answers are the same.

We should also consider exactly what is meant by the expression "current enclosed by the path" . Suppose we solder a circuit together after passing the conductor once through a rubber band , which we shall use to represent the closed path. Some strange and formidable paths can be constructed by twisting and knotting the rubber band , but if neither the rubber band nor the conducting circuit is broken , the current enclosed by the path is that carried by the conductor. Now let us replace the rubber band by a circular ring of spring steel across which is stretched a rubber sheet. The steel loop forms the closed path, and the current conductor must pierce the rubber sheet if the current is to be enclosed by the path. Again, we may twist the steel loop, and we may also deform the rubber sheet by pushing our fist into it or folding it in any way we wish. A single current carrying conductor still pierces the sheet once, and this is the true measure of the current enclosed by the path. If we thread the conductor once through the sheet from front to back and once from back to front , the total current enclosed by the path is the algebraic sum, which is zero.

In more general language, given a closed path, we recognize this path as the perimeter of an infinite number of surfaces (not closed surfaces). Any current carrying conductor enclosed by the path must pass through every one of these surfaces once. certainly some of the surfaces may be chosen in such a way that the conductor pierces them twice in one direction and once in the other direction, but the algebraic sum of the current is still the same.

We shall find that the closed path is usually of an extremely simple nature and can be drawn on a plane. We need merely find the total current passing through this region of the plane.

7.0.1 APPLICATIONS:

7.0.1.1 INFINITELY LONG FILAMENT:

Consider an infinitely long filamentary conductor carrying a current I A . The filament lies on the $z-$ axis in free space, and the current flows in the direction of a_z. Symmetry shows that there is no variation with z or ϕ . Using Biot-Savart law, the direction of dH is perpendicular to the plane containing dL and R and therefore is in the direction of a_ϕ . Hence the only component of $H$is H_ϕ and it is a function only of ρ .

The path chose should be a circle of radius ρ and Ampere's circutal law becomes

$$\oint H \bullet dl = \int_0^{2\pi} H_\phi \rho d\phi = H_\phi 2\pi\rho = I \tag{7.2}$$

or

$$H_\phi = \frac{I}{2\pi\rho} \tag{7.3}$$

as before.

7.0.1.2 INFINITELY LONG COAXIAL TRANSMISSION LINE:

Consider an infinitely long coaxial transmission line carrying a uniformly distributed total current I in the center conductor and $-I$ in the outer conductor. The line is shown in the figure below.

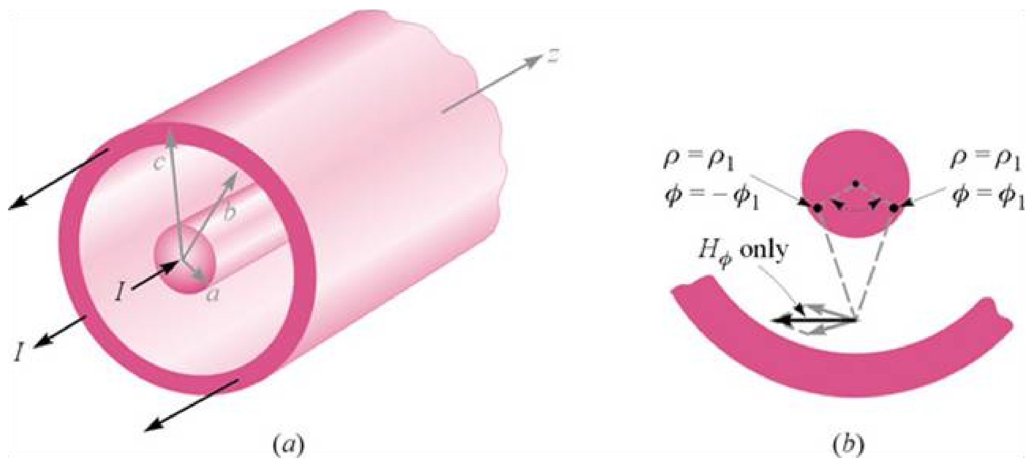


Figure 7.2: Coaxial transmission Line

Symmetry shows that H is not a function of ϕ or z . In order to determine the components present, we may use the results of the previous example by considering solid conductors as being composed of large number of filaments. No filament has a z component of H . Furthermore, the H_ρ component at $\phi = 0$, produced by one filament located at $\rho = \rho_1, \phi = \phi_1$, is canceled by the H_ρ component produced by a symmetrically located at $\rho = \rho_1$, $\phi = -\phi_1$. So only an H_ϕ component which varies with ρ remains.

A circular path of radius ρ , where ρ is larger than the radius of the inner conductor but less than the inner radius of the outer conductor, then leads immediately to

$$H_\phi = \frac{I}{2\pi\rho} \ \ (a < \rho < b) \tag{7.4}$$

If we chose ρ smaller than the radius of the inner conductor, the current enclosed is

$$I_{encl} = I\frac{\rho^2}{a^2} \tag{7.5}$$

and

$$2\pi\rho H_\phi = I\frac{\rho^2}{a^2} \tag{7.6}$$

or

$$H_\phi = \frac{I\rho}{2\pi a^2} \ \ (\rho < a) \tag{7.7}$$

If the radius ρ is larger than the outer radius of the outer conductor , no current is enclosed and

$$H_\phi = 0 \ (\rho > c) \tag{7.8}$$

Finally, if the path lies within the outer conductor , we have

$$\begin{aligned} 2\pi\rho H_\phi &= I - I\left(\frac{\rho^2 - b^2}{c^2 - b^2}\right) \\ H_\phi &= \frac{I}{2\pi\rho}\frac{c^2 - \rho^2}{c^2 - b^2} \ \ (b < \rho < c) \end{aligned}$$

The magnetic field strength variation with radius is shown in the figure given below.

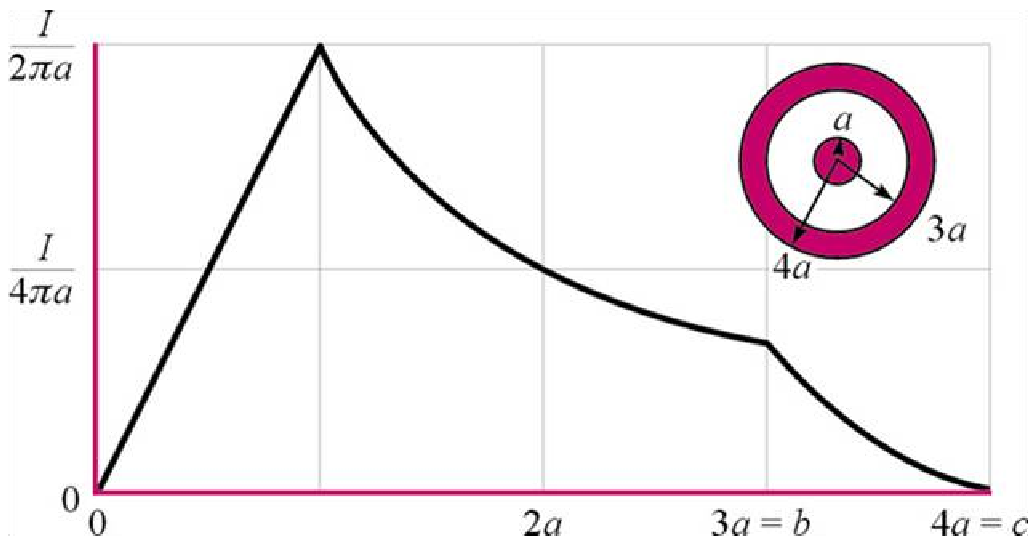


Figure 7.3: Magnetic Field Intensity For a coaxial cable

Foe the coaxial cable $b = 3a$ and $c = 4a$. It should be noted that the magnetic field intensity H is continuous at all conductor boundaries. In other words, a slight increase in the radius of the closed path does not result in the encloser of a tremendously different current. The value of H_ϕ shows no sudden jumps.

The external field is zero. This, we see, results from equal positive and negative currents enclosed by the path. Each produces an external field of magnitude $\frac{I}{2\pi\rho}$, but complete cancellation occurs. This is another example of shielding. Such a coaxial cable carrying large currents would not produce any noticeable effect in an adjacent circuit.

7.0.2 AMPERE'S CIRCUITAL LAW AND MAXWELL'S EQUATION:

Ampere's circuital law is given by

$$\oint H \bullet dl = I_{enc} \tag{7.9}$$

Ampere's law is similar to Gauss's law and it is easily applied to determine H when the current distribution is symmetrical. It should be noted that the above equation always holds whether the current distribution is symmetrical or not but we can only use the equation to determine H when symmetrical current distribution exists. Ampere's law is a special case of Biot-Savart law; the former may be derived from the latter.

By applying Stoke's theorem to the left hand side of the above equation , we obtain

$$I_{enc} = \oint_L H \bullet dl = \int_s (\nabla \times H) \bullet ds \text{ but } I = \int_{encs} J \bullet ds \tag{7.10}$$

Comparing the surface integrals , we get

$$\nabla \times H = J \tag{7.11}$$

This is third of the Maxwell's equations. It is essentially Ampere's law in differential or point form. As the curl of J is not equal to zero, the magnetostatic field is not a conservative field. The integral form of the equation is given by

$$\oint_L \mathbf{H} \bullet dl = \int_s J \bullet ds \tag{7.12}$$

The differential and integral forms of the Maxwell's equation for static magnetic field is given by

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{7.13}$$

$$\oint_L \mathbf{H} \bullet \mathbf{dl} = \int \mathbf{J} \bullet \mathbf{ds} \tag{7.14}$$

(7.15)

7.0.2.1 Applications Of Ampere's Circuital Law

We now apply Ampere's circuital law to determine **H** for symmetrical current distributions as we did for Gauss's law. We will consider an infinite line current , an infinite current sheet.

Infinite Line current:

Consider an infinitely long filamentary current I along the $z-$ axis as in the figure below. To determine **H** at an observation point P , we allow a closed path pass through P .This path, on which Ampere's law is to be applied is known as *Amperian path*. We choose a concentric circle as the Amperian path in view of the nature of the problem which says that **H** is constant provided ρ is constant . Since this path encloses the whole current I , according to Ampere's law

$$I = \int H_\phi a_\phi \bullet \rho d\phi a_\phi = H_\phi \int \rho d\phi = H_\phi.2\pi\rho \tag{7.16}$$

or

$$\mathbf{H} = \frac{I}{2\pi\rho} a_\phi \tag{7.17}$$

Infinite Sheet Of Current:

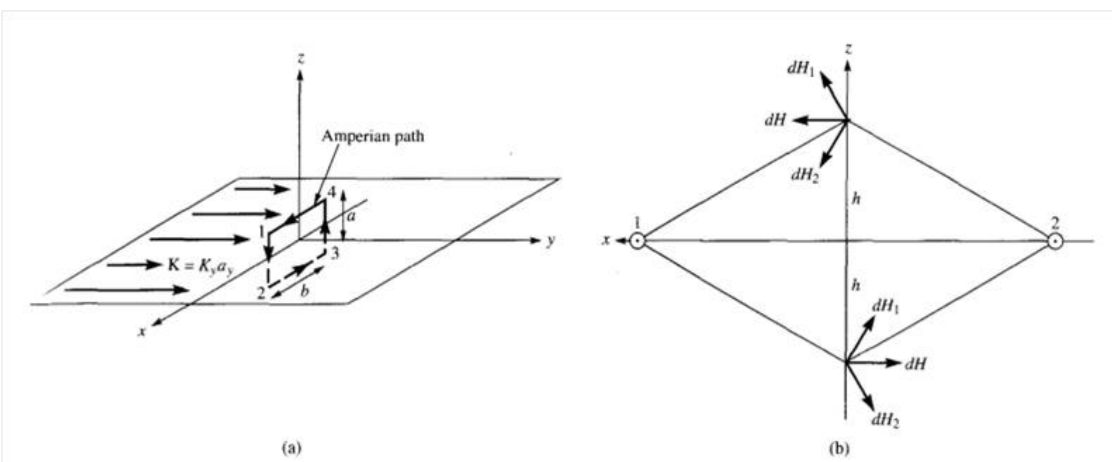
Consider an infinite sheet in the $z = 0$ plane. If the sheet has a uniform current density $\mathbf{k} = k_y a_y$ A/m as shown in figure, then applying Ampere's circuital law to the rectangular closed path (Amperian path) gives

$$\oint \mathbf{H} \bullet dl = I_{enc} = k_y b \qquad (7.18)$$

To evaluate the integral , we first need to have an idea of what **H** is like.To achieve this, we regard the infinite sheet as comprising of filaments; $d\mathbf{H}$ above or below the sheet due to a pair of filamentary currents can be found . As evident in the figure , the resultant $d\mathbf{H}$ has only an $x-$ component. Also , **H** one side of the sheet is the negative of that on the other side . Due to the infinite extent of the sheet , the sheet can be regarded as consisting of such filamentary pairs so that the characteristics of **H** for a pair are the same for the infinite current sheets, that is

$$\mathbf{H} = \begin{cases} H_0 a_x & z > 0 \\ -H_0 a_x & z < 0 \end{cases} \qquad (7.19)$$

where H_0 is yet to be determined . Evaluating the line integral of **H** along the closed path gives



$$\oint \mathbf{H} \bullet dl = \left(\int_1^2 + \int_2^3 + \int_3^4 + \int_4^1 \right) \mathbf{H} \bullet dl$$
$$= 0(-a) + (-H_0)(-b) + 0(a) + H_0(b)$$
$$= 2H_0 b$$

From the above relations , we obtain $H_0 = \frac{1}{2}k_y$. Substituting

$$\mathbf{H} = \begin{cases} \frac{1}{2}k_y a_x & z > 0 \\ -\frac{1}{2}k_y a_x & z > 0 \end{cases} \tag{7.20}$$

In general , for an infinite sheet of current density $\mathbf{k}$ A/m

$$\mathbf{H} = \frac{1}{2}\mathbf{k} \times \mathbf{a_n} \tag{7.21}$$

where $\mathbf{a_n}$is the unit normal vector directed from the current sheet to the point of interest.

Unit-VI

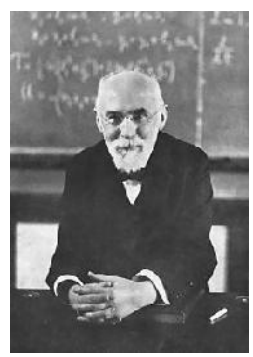
Force in magnetic fields:

Magnetic force - Moving charges in a Magnetic field – Lorentz force equation – force on a current element in a magnetic field – Force on a straight and a long current carrying conductor in a magnetic field – Force between two straight long and parallel current carrying conductors – Magnetic dipole and dipole moment – a differential current loop as a magnetic dipole – Torque on a current loop placed in a magnetic field

Chapter 8

MAGNETIC FORCES, MATERIALS, AND INDUCTANCE

Hendrik Antoon Lorentz (Arnhem, 18 July 1853 – Haarlem, 4 February 1928) was a Dutch physicist who shared the 1902 Nobel Prize in Physics with Pieter Zeeman for the discovery and theoretical explanation of the Zeeman effect. He also derived the transformation equations subsequently used by Albert Einstein to describe space and time.



Hendrik Lorentz was born in Arnhem, Gelderland (The Netherlands), the son of Gerrit Frederik Lorentz (1822–1893), a well-off nurseryman, and Geertruida van Ginkel (1826–1861). In 1862, after his mother's death, his father married Luberta Hupkes. Despite being raised as a Protestant, he was a freethinker in religious matters.[B 1] From 1866 to 1869 he attended the newly established high school in Arnhem, and in 1870 he passed the exams in classical languages which were then required for admission to University.

Lorentz studied physics and mathematics at the University of Leiden, where he was strongly influenced by the teaching of astronomy professor Frederik Kaiser; it was his influence that led him to become a physicist.

8.0.1 FORCE ON A MOVING CHARGE:

In an electric field the definition of the field intensity shows us that the force on a charged particle is

$$F = QE \tag{8.1}$$

The force is in the same direction as the electric field intensity (for a positive charge) and is directly proportional to both E and Q . If the charge is in motion, the force at any point in its trajectory is given by the above equation.

A charged particle in motion in a magnetic field of flux density B is found experimentally to experience a force whose magnitude is proportional to the product of the magnitudes of the charge Q , its velocity v , and the flux density B , and to the sine of the angle between the vectors v and B . The direction of the force is is perpendicular to both v and B and is given by the unit vector in the direction of $v \times B$. The force therefore is expressed as

$$F = Qv \times B \tag{8.2}$$

The force on a moving particle due to combined electric and magnetic fields is obtained by superposition as

$$F = Q(E + v \times B) \tag{8.3}$$

The equation is known as the Lorentz's force equation, and its solution is required in determining electron orbits in the magnetron, proton paths in the cyclotron, plasma characteristics in a magnetohydrodynamic (MHD) generator, or , in general , charged particle motion in combined electric and magnetic fields.

8.0.2 FORCE ON A DIFFERENTIAL CURRENT ELEMENT:

The force on a charged particle moving through a steady magnetic field may be written as the differential force exerted on a differential element of charge,

$$dF = dQv \times B \tag{8.4}$$

The differential element of charge may also be expressed in terms of volume charge density ,

$$dQ = \rho_v dv \tag{8.5}$$

thus

$$dF = \rho_v dv\, v \times B \tag{8.6}$$

or

$$dF = J \times Bdv \tag{8.7}$$

but we know that $Jdv = Kds = IdL$ and thus the Lorentz's force equation may be applied to surface current density

$$dF = K \times Bds \tag{8.8}$$

or to a differential current filament

$$dF = IdL \times B \tag{8.9}$$

Integrating the above equations over a volume, surface, or a closed path , respectively , leads to the integral formulations

$$F = \int_{vol} J \times Bdv$$

$$F = \int_{s} K \times Bds$$

and

$$F = \oint_l I dL \times B = -I \oint_l B \times dL \tag{8.10}$$

A simple result is obtained by applying the last equation to a straight conductor in a uniform magnetic field

$$F = IL \times B \tag{8.11}$$

The magnitude of the force is given by the familiar equation

$$F = BIL \sin\theta \tag{8.12}$$

where θ is the angle between the vectors representing the direction of current flow and the direction of the magnetic flux density.

Example:

Consider the figure below.

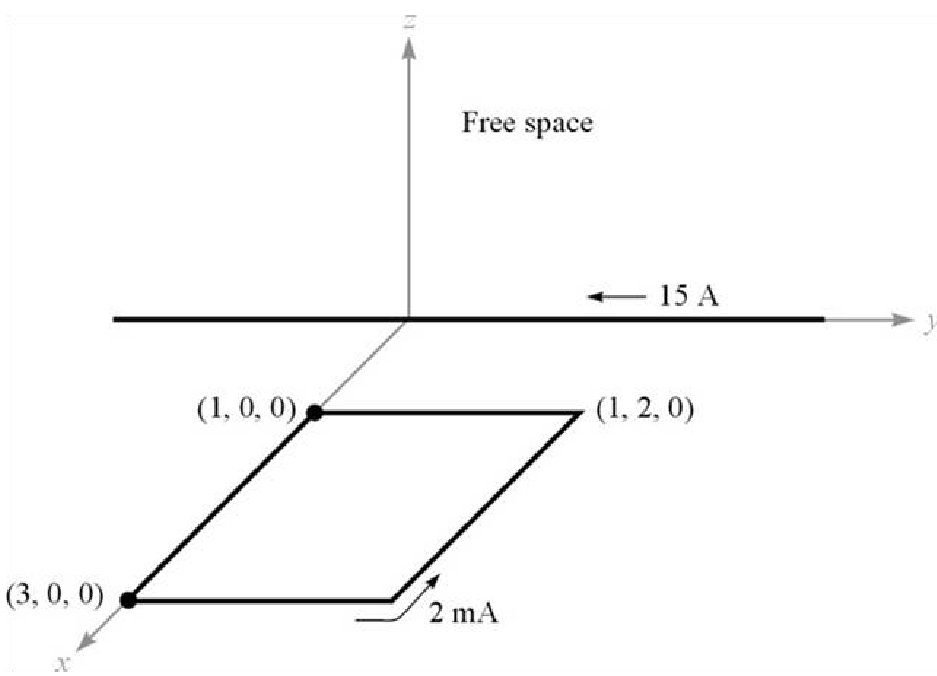


Figure 8.1: Square Loop Of Wire In The xy-plane

We have a square loop of wire in the $z = 0$ plane carrying $2\,mA$ of current in the field of an infinitely long filament on the $y - axis$. The field produced in the plane of the loop by the straight filament is

$$H = \frac{I}{2\pi x} a_z = \frac{15}{2\pi x} a_z \ \ A/m \tag{8.13}$$

therefore

$$B = \mu_0 H = 4\pi \times 10^{-7} H = \frac{3 \times 10^{-6}}{x} a_z \ \ T \tag{8.14}$$

Then

$$F = -\oint B \times dL \tag{8.15}$$

Let us assume a rigid loop so that the total force is the sum of the forces on the four sides. Beginning with the left side:

$$\begin{aligned} F &= -2 \times 10^{-3} \times 3 \times 10^{-6} \left[\int_{x=1}^{3} \frac{a_z}{x} \times dx a_x + \int_{y=1}^{3} \frac{a_z}{3} \times dy a_y + \int_{x=3}^{1} \frac{a_z}{x} \times dx a_x + \int_{y=3}^{1} \right. \\ F &= -6 \times 10^{-9} \left[\ln|_1^3 \, a_y + \frac{1}{3} y \Big|_1^3 (-a_x) + \ln x|_3^1 \, a_y + y|_3^1 (-a_x) \right] \\ F &= -6 \times 10^{-9} \left[(\ln 3) a_y - \frac{2}{3} a_x + \left(\ln \frac{1}{3} \right) a_y + 2 a_x \right] = -8 a_x \ pN \end{aligned}$$

Thus the net force on the loop is in the $-a_x$ direction.

8.0.3 FORCE BETWEEN DIFFERENTIAL CURRENT ELEMENTS:

It is possible to express the force on one current element directly in terms of a second current element without finding the magnetic field. The magnetic field at point 2 due to a current element at point 1 has been found to be

$$dH_2 = \frac{I_1 dL_1 \times a_{R_{12}}}{4\pi R_{12}^2} \quad (8.16)$$

Now, the differential force on a differential current element is

$$dF = IdL \times B \quad (8.17)$$

and we apply this to our problem by letting B be dB_2 (the differential flux density at point 2 caused by current element at 1) , by identifying IdL as I_2dL_2 , and by symbolizing the differential of our differential force on element 2 as $d(dF_2)$

$$d(dF_2) = I_2 dL_2 \times dB_2 \quad (8.18)$$

since $dB_2 = \mu_0 dH_2$, we obtain the force between two differential current elements ,

$$d(dF_2) = \mu_0 \frac{I_1 I_2}{4\pi R_{12}^2} dL_2 \times (dL_1 \times a_{R_{12}}) \quad (8.19)$$

Example:

Consider two differential current elements shown in the figure. $I_1dL_1 = -3a_y$ A.m at $P_1(5, 2, 1)$, and $I_2dL_2 = -4a_z$ A.m at $P_2(1, 8, 5)$.

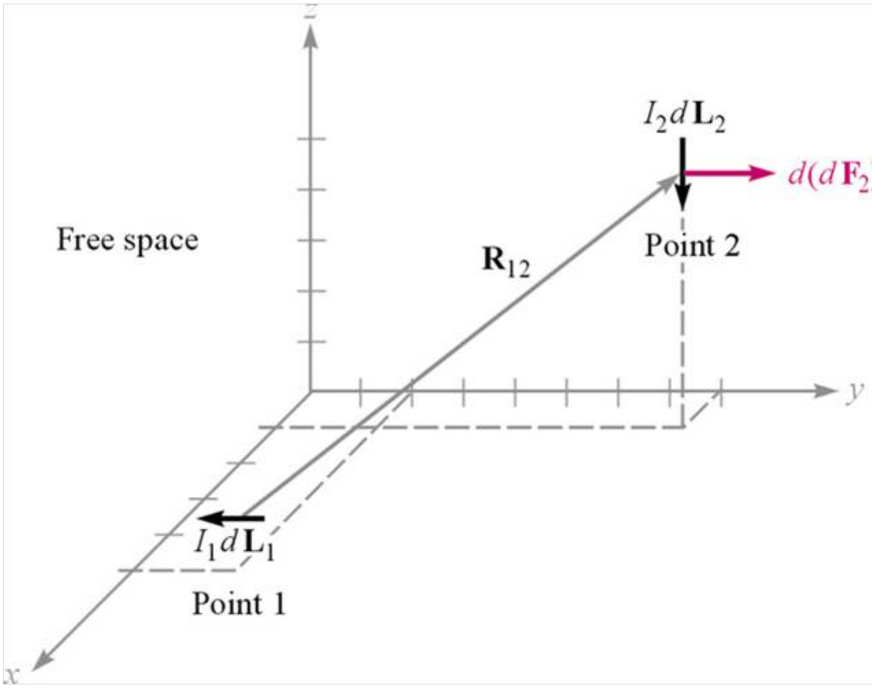


Figure 8.2: Force Between Two Current Elements

Thus $R_{12} = -4a_x + 6a_y + 4a_z$, and we may substitute this data in the equation resulting in

$$d(dF_2) = \frac{4\pi \times 10^{-7}}{4\pi} \frac{(-4a_z) \times [(-3a_y) \times (-4a_x + 6a_y + 4a_z)]}{(16 + 36 + 16)^{1.5}} = 8.56a_y \text{ nN} \tag{8.20}$$

If we find $d(dF_1)$ it is equal to $-12.8a_z$ nN which is not equal and opposite to the force $d(dF_2)$. The reason for this is that a differential current element is an abstraction, and it can not exist in practice. The continuity of current demands that a complete circuit be considered .

So the total force between two filamentary circuits is obtained by integrating twice:

$$F_2 = \mu_0 \frac{I_1 I_2}{4\pi} \oint \left[dL_2 \times \oint \frac{dL_1 \times a_{R_{12}}}{R_{12}^2} \right]$$

$$F_2 = \mu_0 \frac{I_1 I_2}{4\pi} \oint \left[\oint \frac{a_{R_{12}} \times dL_1}{R_{12}^2} \right] \times dL_2$$

The above equation, though appears to be formidable, it is not difficult to use. It can be used to find the force between two infinitely long, straight , parallel, filamentary conductors with separation d , and carrying equal but opposite currents.

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The magnetic field intensity at either wire caused by the other is already known to be $\frac{I}{2\pi d}$. it can be seen that the force is

$$\mu_0 \frac{I^2}{2\pi d} \text{ newtons per meter length.} \tag{8.21}$$

This can be derived in a different way as shown below

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$$
\begin{aligned}
F_1 =& \frac{\mu_0 I_1 I_2}{4\pi} \oint \oint \frac{dl_1 \times (dl_2 \times a_R)}{R^2} N \\
=& \oint \oint \frac{(\mu_0 I_1 dl_1)(I_2 dl_2 \cos\theta)}{4\pi R^2} \\
R = d\sec\theta, l_2 = d\tan\theta,\ & dl_2 = d\sec^2\theta d\theta \\
=& \frac{\mu_0 I_1 I_2}{4\pi d} \int_0^1 dl_1 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta d\theta \\
F_1 =& \frac{\mu_0 I_1 I_2}{2\pi d} N/m
\end{aligned}
$$

8.0.4 FORCE AND TORQUE ON A CLOSED CIRCUIT:

The force on a filamentary closed circuit is given by

$$F = -I \oint B \times dL \tag{8.22}$$

If we assume that the magnetic field is uniform , then B can be removed from the integral

$$F = -IB \times \oint dL \tag{8.23}$$

but the closed line integral $\oint dL = 0$. Therefore the force on a closed filamentary circuit in a uniform magnetic field is zero. If the field is not uniform , the total force need not be zero.

Although the force is zero, the torque is generally not zero. In determining the torque , or moment , of a force , it is necessary to consider both an origin at or about which the torque is to be calculated as well as the point at which the force is applied. See the figure below:

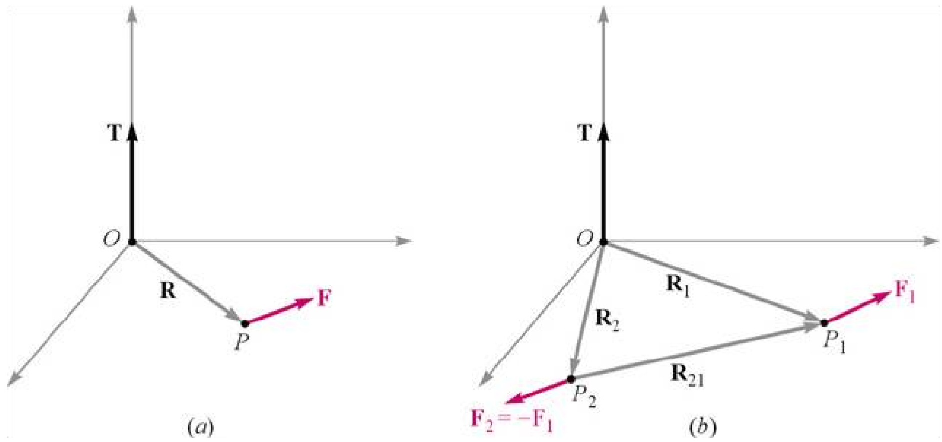


Figure 8.3:

We apply a force F at point P , and we establish an origin at O with a rigid lever arm Rextending from O to P . The torque about point O is a vector whose magnitude is the product of the magnitudes of R and F , and of the sine of the angle between these two vectors. The direction of the vector torque T is normal to both the force F and lever arm R and is in the direction of progress of a right handed screw as the lever arm is rotated into the force vector through the smaller angle . The torque is expressible as a cross product

$$T = R \times F \tag{8.24}$$

Now let us assume that two forces , F_1 at P_1 and F_2 at P_2 , having lever arms R_1 and R_2 extending from a common origin O , as shown in the figure are applied to an object of fixed shape and that the object does not undergo any translation. The torque about the origin is

$$T = R_1 \times F_1 + R_2 \times F_2 \tag{8.25}$$

where

$$F_1 + F_2 = 0 \tag{8.26}$$

and therefore

$$T = (R_1 - R_2) \times F_1 = R_{21} \times F_1 \tag{8.27}$$

The vector $R_{21} = R_1 - R_2$ joins the point of application of F_2 to that of F_1 and is independent of the choice of origin for the two vectors R_1 and R_2 . Therefore the torque is also independent of the choice of origin, provided that the total force is zero.

We may therefore choose the most convenient origin , and this is usually on the axis of rotation and in the plane containing the applied forces if the several forces are coplanar.

8.0.4.1 TORQUE ON A DIFFERENTIAL CURRENT LOOP:

Consider that a differential current loop carrying a current I is placed in a magnetic field B . Assume that the loop lies in the $xy-$ plane .

The sides of the loop are parallel to the x and y axes and are of length dx and dy . The value of the magnetic field at the center of the loop is taken as B_0 Since the loop is of differential size, the value of B at all points on the loop may be taken as B_0 . The total force on the loop is therefore zero, and we are free to choose the center of the loop for calculation of torque .

The vector force on side 1 is

$$dF_1 = Idxa_x \times B_0 \tag{8.28}$$

where

$$B_0 = B_{0x}a_x + B_{0y}a_y + B_{0z}a_z \tag{8.29}$$

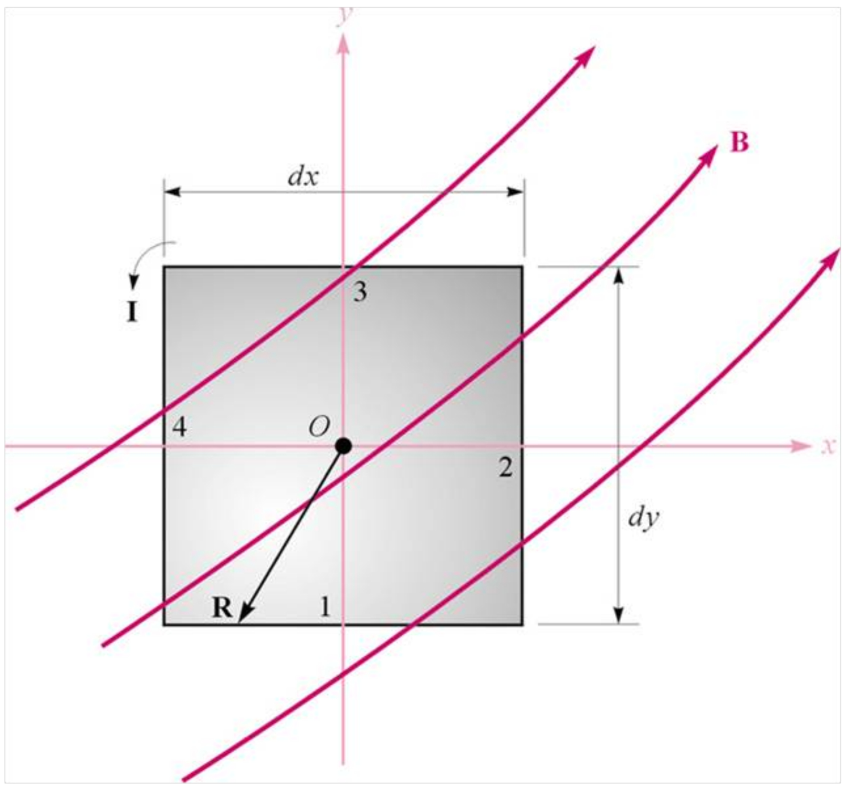


Figure 8.4: Differential Current Loop

so

$$dF_1 = Idx\left(B_{0y}a_z - B_{0z}a_y\right) \tag{8.30}$$

For this side of the loop the lever arm R extends from the origin to the midpoint of the side , $R_1 = -\frac{1}{2}dya_y$, and the contribution to the total torque is

$$\begin{aligned} dT_1 =& R_1 \times dF_1 \\ =& -\frac{1}{2}dya_y \times Idx\left(B_{0y}a_z - B_{0z}a_y\right) \\ =& -\frac{1}{2}dxdyIB_{0y}a_x \end{aligned}$$

The torque contribution on side 3 is found to be the same

$$\begin{aligned} dT_3 =& R_3 \times dF_3 \\ =& \frac{1}{2}dya_y \times (-Idxa_x \times B_0) \\ =& -\frac{1}{2}dxdyIB_{0y}a_x = dT_1 \end{aligned}$$

and

$$dT_1 + dT_3 = -dxdyIB_{0y}a_x \tag{8.31}$$

Evaluating the torque on sides 2 and 4 , we find that

$$dT_2 + dT_4 = dxdyIB_0a_y \tag{8.32}$$

and the total torque is

$$dT = Idxdy(B_{0x}a_y - B_{0y}a_x) \tag{8.33}$$

The quantity within the parenthesis may be represented by a cross product

$$dT = Idxdy(a_z \times B_0) \tag{8.34}$$

or

$$dT = Ids \times B \tag{8.35}$$

where ds is the vector area of the differential current loop and the subscript on B_0 has been dropped. Define the product of the loop current and the vector area of the loop as the magnetic dipole moment dm with units of $a.m^2$. So

$$\begin{aligned} dm &= Ids \\ dT &= dm \times B \end{aligned}$$

We should note that the torque on the current loop always tends to turn the loop so as to align the magnetic field produced by the loop with the applied magnetic field that is causing the torque. This is the easiest way to determine the direction of the torque.

Example:

Consider the rectangular loop shown. The loop has dimensions of $1m$ by $2m$ and lies in the uniform field

$$B_0 = -0.6a_y + 0.8a_z \; T \tag{8.36}$$

The loop current is $4mA$. Calculate the torque.

Ans:

Let us calculate the torque by using $T = Ids \times B$

$$T = 4 \times 10^{-3} \left[(1)(2)a_z \times (-0.6a_y + 0.8a_z)\right] = 4.8a_x \; mN.m \tag{8.37}$$

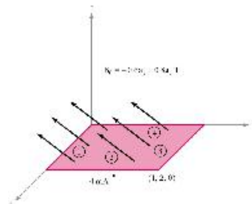


Figure 8.5: Rectangular Loop In A Uniform Field

The loop tends to rotate about an axis parallel to the positive $x-$ axis . The small magnetic field produced by the $4-mA$ current tends to line up with B_0

8.0.5 Magnetization in Materials:

Without an external **B** field applied to the material, the sum of **m**'s is zero due to the random orientations. When an external **B** field is applied , the magnetic moments of the electrons more or less align themselves with the **B** so that the net magnetic moment is not zero.

The magnetization **M** (in A/m) is the dipole moment /unit volume.

If there are N atoms in a given volume Δv and the $k\,th$ atom has a magnetic moment $\mathbf{m_k}$

$$\mathbf{M} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{N} \mathbf{m_k}}{\Delta v} \tag{8.38}$$

A medium for which **M** is not zero everywhere is said to be magnetized. The vector magnetic potential due to $d\mathbf{m}$ is

$$d\mathbf{A} = \frac{\mu_0 \mathbf{M} \times a_R}{4\pi R^2} dv' = \frac{\mu_0 \mathbf{M} \times R}{4\pi R^3} dv' \tag{8.39}$$

$$\frac{R}{R^3} = \nabla' \left(\frac{1}{R} \right) \tag{8.40}$$

Hence

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int \mathbf{M}\times\nabla'\left(\frac{1}{R}\right)dv' \tag{8.41}$$

$$\mathbf{M}\times\nabla'\left(\frac{1}{R}\right) = \left(\frac{1}{R}\right)\nabla'\times\mathbf{M} - \nabla'\times\left(\frac{\mathbf{M}}{R}\right) \tag{8.42}$$

So

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_{v'}\left(\frac{1}{R}\right)\nabla'\times\mathbf{M}dv' - \frac{\mu_0}{4\pi}\int_{v'}\nabla'\times\frac{\mathbf{M}}{R}dv' \tag{8.43}$$

From the vector identity

$$\int_{v'}\nabla'\times\mathbf{F}dv' = -\oint_{s'}\mathbf{F}\times ds \tag{8.44}$$

we can rewrite the expression for $\mathbf{A}$ as

$$\begin{aligned}\mathbf{A} &= \frac{\mu_0}{4\pi}\left[\int_{v'}\frac{\nabla'\times\mathbf{M}}{R}dv' + \oint_{s'}\frac{\mathbf{M}\times a_n}{R}ds'\right]\\ &= \frac{\mu_0}{4\pi}\left[\int_{v'}\frac{\mathbf{J_b}}{R}dv' + \oint_{s'}\frac{\mathbf{K_b}}{R}ds'\right]\\ \mathbf{J_b} &= \nabla\times\mathbf{M}\quad \mathbf{K_b} = \mathbf{M}\times a_n\end{aligned}$$

where $\mathbf{J_b}$ = Bound current density and $\mathbf{K_b}$ = Bound surface current density . In free space $\mathbf{M} = 0$ and we have

$$\nabla\times\mathbf{H} = \mathbf{J_f} \tag{8.45}$$

or

$$\nabla\times\frac{\mathbf{B}}{\mu_\mathbf{0}} = \mathbf{J_f} \tag{8.46}$$

where $\mathbf{J_f}$ is the free volume current density. In material medium $\mathbf{M} \neq 0$, and as a result $\mathbf{B}$ changes

$$\nabla \times \frac{\mathbf{B}}{\mu_0} = \mathbf{J_f} + \mathbf{J_b} = \mathbf{J}$$
$$= \nabla \times \mathbf{H} + \nabla \times \mathbf{M}$$

or

$$\mathbf{B} = \mu_0 \left(\mathbf{H} + \mathbf{M} \right) \tag{8.47}$$

but

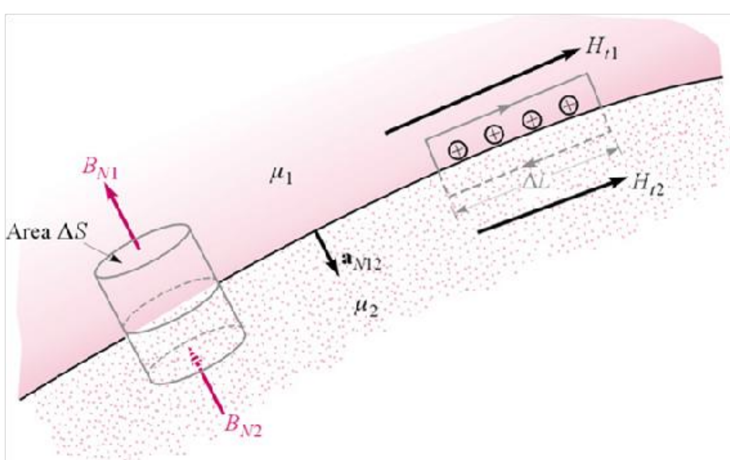
$$\mathbf{M} = \chi_m \mathbf{H}, \quad \mathbf{B} = \mu_\mathbf{0} \left(\mathbf{1} + \chi_\mathbf{m} \right) \mathbf{H} = \mu_\mathbf{0} \mu_\mathbf{r} \mathbf{H} \tag{8.48}$$

$$\mu_r = 1 + \chi_m = \frac{\mu}{\mu_0} \tag{8.49}$$

$\mu = \mu_0 \mu_r$ and is called the permeability of the material.

8.0.6 Magnetic Boundary Conditions:

Figure below shows a boundary between two isotropic materials with permeabilities μ_1 and μ_2 . The boundary condition on the normal components is determined by allowing the surface to cut small cylindrical Gaussian surface.



Magnetic Boundary Conditions

Applying Gauss law for the magnetic field

$$\oint_s \mathbf{B} \bullet ds = 0 \tag{8.50}$$

we find that

$$B_{N1} \triangle s - B_{N2} \triangle s = 0 \tag{8.51}$$

or

$$B_{N1} = B_{N2} \tag{8.52}$$

Thus

$$H_{N2} = \frac{\mu_1}{\mu_2} H_{N1} \tag{8.53}$$

The normal component of **B** is continuous , but the normal component of **H** is discontinuous by the ratio $\frac{\mu_1}{\mu_2}$. The relationship between the normal components of **M** , of course is fixed once the relationship between the normal components of **H** is known. The result is

$$M_{N2} = \frac{\chi_{m2} \mu_1}{\chi_{m1} \mu_2} \tag{8.54}$$

Next, apply Amper's circuital law to the rectangular loop

$$\oint \mathbf{H} \bullet dl = I \tag{8.55}$$

Taking a clockwise trip along the loop we fin that

$$H_{t1} \triangle l - H_{t2} \triangle l = K \triangle l \tag{8.56}$$

where we assume that the boundary may carry a current **K** whose component normal to the plane of the closed path is K. Thus

(8.57)

Unit-VII

Magnetic Potential:

Scalar Magnetic potential and its limitations – vector magnetic potential and its properties – vector magnetic potential due to simple configurations – vector Poisson's equations. Self and Mutual inductance – Neumans's formulae – determination of self-inductance of a solenoid and toroid and mutual inductance between a straight long wire and a square loop wire in the same plane – energy stored and density in a magnetic field. Introduction to permanent magnets, their characteristics and applications.

Chapter 9

Magnetic potential

Joseph Henry (December 17, 1797 – May 13, 1878) was an American scientist who served as the first Secretary of the Smithsonian Institution, as well as a founding member of the National Institute for the Promotion of Science, a precursor of the Smithsonian Institution.[1] During his lifetime, he was highly regarded. While building electromagnets, Henry discovered the electromagnetic phenomenon of self-inductance. He also discovered mutual inductance independently of Michael Faraday, though Faraday was the first to publish his results.[2][3] Henry was the inventor of the electric doorbell (1831)[4] and relay (1835).[5] The SI unit of inductance, the henry, is named in his honor. Henry's work on the electromagnetic relay was the basis of the electrical telegraph, invented by Samuel Morse and Charles Wheatstone separately.

9.1 Scalar magnetic potential:

In electrostatics ,$\nabla \times \mathbf{E} = 0$. So $\mathbf{E}$ is expressed as $-\nabla V$, where V is a scalar potential. This is a stepping stone which allows solving problems using several small steps.

In magnetic fields , $\mathbf{H}$ can also be expressed as a gradient of a scalar magnetic potential . So

$$\mathbf{H} = -\nabla V_m \tag{9.1}$$

The selection of $-$ve gradient will provide us with a clear analogy to the electrical potential. The above definition should not conflict with our previous results

$$\nabla \times \mathbf{H} = \mathbf{J} = \nabla \times (-\nabla V_m) \tag{9.2}$$

but curlgrad of any scalar is zero(vector identity). So if $\mathbf{H}$ is to be defined as the gradient of a scalar , then current density must be zero throughout the region in which the scalar magnetic potential is so defined.

$$\mathbf{H} = -\nabla V_m \ \ (\mathbf{J} = 0) \tag{9.3}$$

Many magnetic problems involve geometries in which the current carrying conductors occupy a relatively small fraction of the total region of interest. The dimensions of V_m are Ampere.

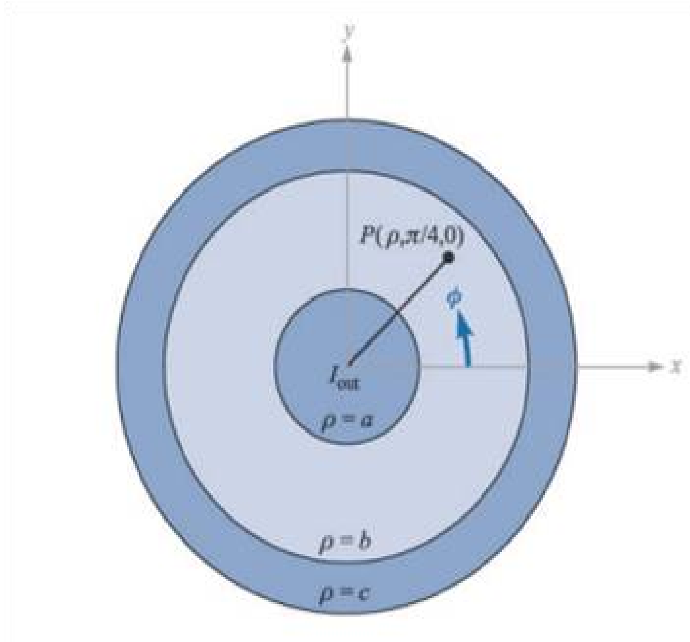
In free space

$$\begin{aligned} \nabla \bullet \mathbf{B} =& \mu_0 \nabla \bullet \mathbf{H} = 0 \\ \mu_0 \nabla \bullet (-\nabla V_m) =& 0 \\ \nabla^2 V_m =& 0 \ \ (\mathbf{J} = 0) \end{aligned}$$

Unlike electrostatic potential V_m is not a single valued function of position.

Example:

Consider the cross section of the co-axial line shown.



In the region $a < \rho < b$, $\mathbf{J} = 0$. So

$$\mathbf{H} = \frac{I}{2\pi\rho} a_\phi \tag{9.4}$$

I is the current in the a_z direction in the inner conductor.

$$\frac{I}{2\pi\rho} = -\nabla V_m = -\frac{1}{\rho}\frac{\partial V_m}{\partial \phi}$$
$$\frac{\partial V_m}{\partial \phi} = -\frac{I}{2\pi}$$
$$V_m = -\frac{I}{2\pi}\phi$$

where the constant of integration is set to zero. What is the value of V_m at P ? here $\phi = \frac{\pi}{4}$. If V_m be zero at $\phi = 0$ and proceed counter clock-wise around the circle , the magnetic potential goes negative linearly. For a full circle , the potential is $-I$, but that was the point at which the potential is assumed to be zero.

At P, $\phi = \frac{\pi}{4}, \frac{9\pi}{4}, \frac{17\pi}{4}$ or $-\frac{7\pi}{4}, -\frac{23\pi}{4}$.....$etc$

$$V_{mP} = \frac{I}{2\pi}\left(2n - \frac{1}{4}\right)\pi, \ (n = 0, \pm 1, \pm 2, ...) \tag{9.5}$$

The reason for this is

$$\nabla \times \mathbf{E} = 0$$

$$\oint_l \mathbf{E} \bullet dl = 0$$

$$V_{mab} = -\int_b^a \mathbf{H} \bullet dl \ \text{(where a specific path is to be selected)}$$

Magnetic scalar potential is not a conservative field.

9.1.1 Vector Magnetic Potential:

we know that for a magnetic field

$$\nabla \bullet \mathbf{B} = 0 \tag{9.6}$$

From the vector identity

$$\nabla \bullet (\nabla \times \mathbf{A}) = 0 \tag{9.7}$$

So if the divergence of a vector field is zero, then this vector can be expressed as the curl of another vector. As the divergence of the magnetic flux density is always zero, the vector **B** can be expressed as the curl of another vector. That is

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{9.8}$$

This can always be done for a divergence free (solenoidal) field . The vector **A** is called the ***vector magnetic potential .***

It is important to note the following:

1. The magnetic vector potential is defined based on the divergence free condition of **B** .

2. The definition of **A** is entirely based on the mathematical properties of the vector **B** , not on its physical characteristics . In this sense , **A** is viewed as an auxiliary function rather than a fundamental field quantity. Nevertheless, the magnetic vector potential is an important function with considerable utility. we will make considerable use of the magnetic vector potential .

3. Since the magnetic vector potential is a vector quantity , both its curl and divergence must be specified. The curl is specified by the above equation. We can safely assume that the divergence is zero $(\nabla \bullet \mathbf{A} = 0)$.

4. The magnetic vector potential does not have a simple physical meaning in the sense that it is not a measurable physical quantity like **B** or **H** . It may seem a bit unsettling to define a physical quantity based on the mathematical properties of another function and then use this secondary function to evaluate physical properties of the magnetic field. In fact there is nothing unusual about this process. We can view the definition of the magnetic vector potential as a transformation. As long as the inverse transformation is unique , there is nothing wrong in **A** not having a readily defined physical meaning. We can use the magnetic vector potential in any way that is consistent with the properties of a vector field and the rules of vector algebra. If we then transform back to the magnetic flux density using the equation $\mathbf{B} = \nabla \times \mathbf{A}$, all results thus obtained are correct.

5. because the magnetic vector potential relates to the magnetic flux density through the curl , the magnetic vector potential **A** is at right angles to the magnetic flux density **B** .

6. The units of **A** are wb/m .

Now we want to get an expression for **A** .

$$\mathbf{B_2} = \nabla_2 \times \mathbf{A_2} = \frac{\mu_0}{4\pi} \oint \frac{I_1 dl_1 \times a_{R_{12}}}{R_{12}^2} = -\frac{\mu_0}{4\pi} \oint \frac{a_{R_{12}} \times I_1 dl_1}{R_{12}^2}$$

$$\text{but } -\frac{a_{R_{12}}}{R_{12}^2} = \nabla_2 \left(\frac{1}{R_{12}}\right)$$

$$\text{so } \mathbf{B_2} = \nabla_2 \times \mathbf{A_2} = \frac{\mu_0}{4\pi} \oint \nabla_2 \left(\frac{1}{R_{12}}\right) \times I_1 dl_1$$

$$\text{to this add } \frac{1}{R_{12}} (\nabla_2 \times I_1 dl_1) = 0$$

$$\mathbf{B_2} = \nabla_2 \times \mathbf{A_2} = \frac{\mu_0}{4\pi} \left[\oint \nabla_2 \left(\frac{1}{R_{12}}\right) \times I_1 dl_1 + \oint \frac{1}{R_{12}} (\nabla_2 \times I_1$$

$$\text{from the identity } \oint \nabla_2 \times \left(\frac{I_1 dl_1}{R_{12}}\right) = \left[\oint \nabla_2 \left(\frac{1}{R_{12}}\right) \times I_1 dl_1 + \oint \frac{1}{R_{12}} (\nabla_2 \times I_1 dl_1$$

$$\text{we can write } \mathbf{B_2} = \nabla_{\mathbf{2}} \times \mathbf{A_2} = \frac{\mu_0}{4\pi} \oint \nabla_2 \times \left(\frac{I_1 dl_1}{R_{12}}\right) = \oint \nabla_2 \times \left(\frac{\mu_0}{4\pi} \frac{I_1 dl_1}{R_{12}}\right)$$

$$B_2 = \nabla_2 \times \mathbf{A_2} = \nabla_2 \times \oint \left(\frac{\mu_0}{4\pi} \frac{I_1 dl_1}{R_{12}}\right)$$

$$\mathbf{A_2} = \frac{\mu_0}{4\pi} \oint \frac{I_1 dl_1}{R_{12}}$$

The significance of the terms in the above equation is the same as in the Biot-Savart law : a direct current I flows along a filamentary conductor of which any differential length $d\mathbf{L}$ is distant R from the point at which **A** is to be found. Since we have defined **A** only through specification of its curl , it is possible to add the gradient of any scalar field to the equation for **A** without changing **B** or **H**, for the the curl of the gradient is identically zero. In steady magnetic fields , it is customary to set this possible added term equal to zero.

Unit-VIII

Time varying fields :

Time varying fields – Faraday's laws of electromagnetic induction – Its integral and point forms – Maxwell's fourth equation, $\nabla \times E = -\frac{\partial B}{\partial t}$ – Statically and Dynamically induced EMFs – Simple problems. Modification of Maxwell's equations for time varying fields – Displacement current – Poynting Theorem and

Chapter 10

FARADAY'S LAW AND ELECTROMAGNETIC INDUCTION

Michael Faraday, FRS (22 September 1791 – 25 August 1867) was an English scientist who contributed to the fields of electromagnetism and electrochemistry. His main discoveries include that of electromagnetic induction, diamagnetism and electrolysis.
Although Faraday received little formal education he was one of the most influential scientists in history,[1] and historians of science[2] refer to him as having been the best experimentalist in the history of science.[3] It was by his research on the magnetic field around a conductor carrying a direct current that Faraday established the basis for the concept of the electromagnetic field in physics. Faraday also established that magnetism could affect rays of light and that there was an underlying relationship between the two phenomena.[4][5] He similarly discovered the principle of electromagnetic induction, diamagnetism, and the laws of electrolysis. His inventions of electromagnetic rotary devices formed the foundation of electric motor technology, and it was largely due to his efforts that electricity became viable for use in technology.

Heinrich Friedrich Emil Lenz (12 February 1804 – 10 February 1865) was a Russian physicist of Baltic German ethnicity. He is most noted for formulating Lenz's law in electrodynamics in 1833. The symbol L, conventionally representing inductance, is chosen in his honor.[1] Lenz was born in Dorpat (now Tartu, Estonia), the Governorate of Livonia, in the Russian Empire at that time. After completing his secondary education in 1820, Lenz studied chemistry and physics at the University of Dorpat. He traveled with the navigator Otto von Kotzebue on his third expedition around the world from 1823 to 1826. On the voyage Lenz studied climatic conditions and the physical properties of seawater. The results have been published in "Memoirs of the St. Petersburg Academy of Sciences" (1831).

After the voyage, Lenz began working at the University of St. Petersburg, Russia, where he later served as the Dean of Mathematics and Physics from 1840 to 1863 and was Rector from 1863 until his death in 1865. Lenz also taught at the Petrischule in 1830 and 1831, and at the Mikhailovskaya Artillery Academy.

Lenz had begun studying electromagnetism in 1831. Besides the law named in his honor, Lenz also independently discovered Joule's law in 1842; to honor his efforts on the problem, it is also given the name the "Joule–Lenz law," named also for James Prescott Joule.

When static conditions hold ie when time does not enter into the picture, electricity and magnetism are two separate , somewhat parallel disciplines. This can be seen by observing that

Maxwell's equations, appear as two sets of equations which are independent of each other ie the two equations describing the electric field has no term which contains a magnetic quantity and the two equations which describe the magnetic field do not contain any electrical field quantity.

$$\nabla \bullet E = \frac{\rho}{\epsilon_0}, \nabla \bullet B = 0$$
$$\nabla \times E = 0, \nabla \times B = \mu_0 J$$

Expressed in terms of the potential fields, these equations are equivalent to

$$\phi = \frac{1}{4\pi\epsilon_0} \int_v \frac{\rho}{R} d\tau,\ A = \frac{\mu_0}{4\pi} \int_v \frac{J}{R} dv$$

ρ is the cause, ϕ and E are the results. J is the cause, A and B are the results. The forces are assumed to be transmitted either with infinite speed or with finite speed such that sufficient time is allowed for an equilibrium situation to develop.

If time varying fields are considered, the equations $\nabla \bullet E$ and $\nabla \bullet B$ remain the same but the other two equations require modification.

When conditions are changing only slowly with respect to time, it is called quasi-static. When conditions are changing rapidly, it is called time varying (radiation effects).

1820 Oerstead demonstrated that an electric current affects a compass needle. In 1831 Faraday showed that a time changing magnetic field will produce an electromotive force.

$$e.m.f = -\frac{d\phi}{dt} \tag{10.1}$$

The change in flux may result from

- A time changing flux linking a stationary path
- Relative motion between flux and a closed path
- Combination of the above two

The first one is called the transformer e.m.f. The second one is called motional e.m.f.

Consider an arrangement of a conducting loop with a galvanometer in the loop and either a permanent magnet or a current carrying coil placed near the conducting loop. Considering various situations the following observations can be made.

1	Place a magnet near the conducting loop	No current flows through the galvanometer
2	Move the magnet towards the loop	The galvanometer registers a current
3	Reverse the direction of motion of the magnet	The galvanometer deflection reverses
4	Reverse the polarity of the magnet and move the magnet	The galvanometer deflection reverses

5	Keep the magnet fixed and move the coil towards the magnet	The galvanometer registers a current
6	Increase the speed of the magnet	The deflection increases
7	Increase the strength of the magnet	The deflection increases
8	Increase the diameter of the coil	The deflection increases
9	Fix the speed of the magnet but repeat with the magnet closer to the coil	The deflection of the galvanometer increases
10	Move the magnet at an angle to the coil	The galvanometer deflection decreases

11	Increase the number of turns of the coil	Magnitude of the current increases

10.0.1 Transformer e.m.f:

If a closed stationary path in space which is linked with a changing magnetic field is considered, it is found that the induced voltage around this path is equal to the negative rate of change of the total flux through the path.

$$\oint E \bullet dl = v_{ind} \tag{10.2}$$

but

$$v_{ind} = -\frac{\partial \phi}{\partial t} = -\frac{\partial}{\partial t}\int_s B \bullet ds \tag{10.3}$$

which results in

$$\oint E \bullet dl = -\frac{\partial}{\partial t}\int_s B \bullet ds \tag{10.4}$$

the figures of our right hand indicate the direction of closed path, and our thumb indicates the direction of ds . A flux density B , in the direction of ds and increasing with time, thus produces an average value of E which is opposite to the positive direction about the closed path. The rigt handed relationship between the surface integral and the closed line integral should always be kept

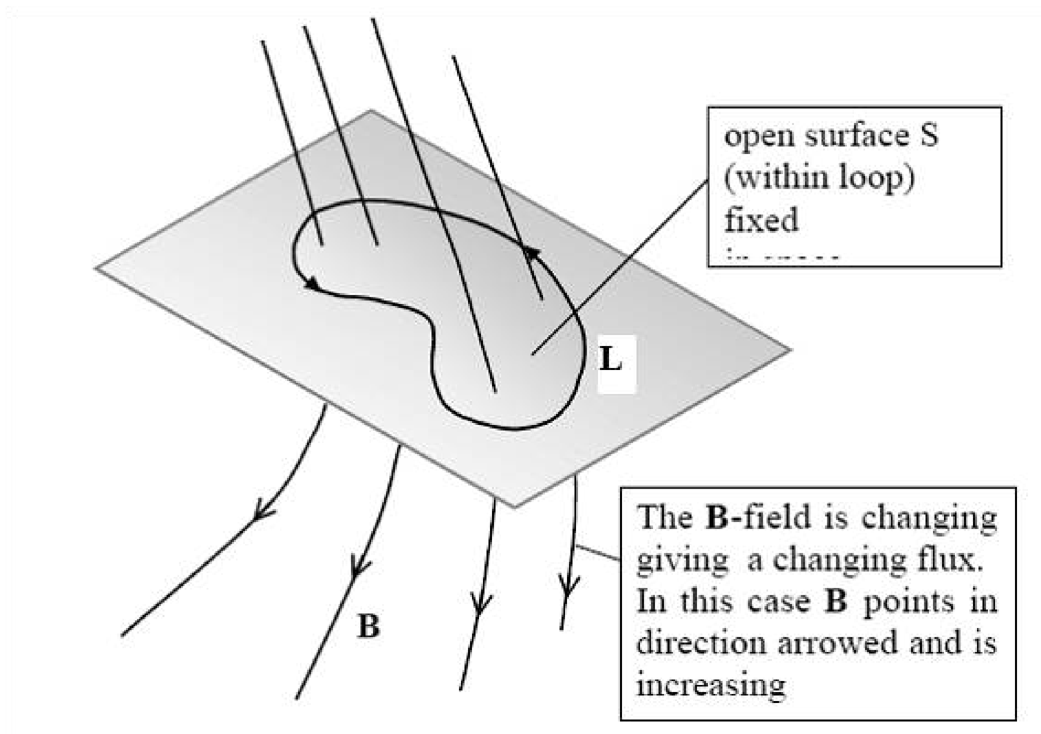


Figure 10.1:

in mind during flux integrations and e.m.f determinations.

$$\oint E \bullet dl = -\frac{\partial}{\partial t} \int_s B \bullet ds \tag{10.5}$$

applying Stoke's theorem

$$\int_s (\nabla \times E) \bullet ds = -\frac{\partial}{\partial t} \int_s B \bullet ds = -\int_s \frac{\partial B}{\partial t} \bullet ds \tag{10.6}$$

which results in

$$\nabla \times E = -\frac{\partial B}{\partial t} \tag{10.7}$$

The e.m.f induced in the loop L defined on the surface S is equal to the rate of change of magnetic flux enclosed by L

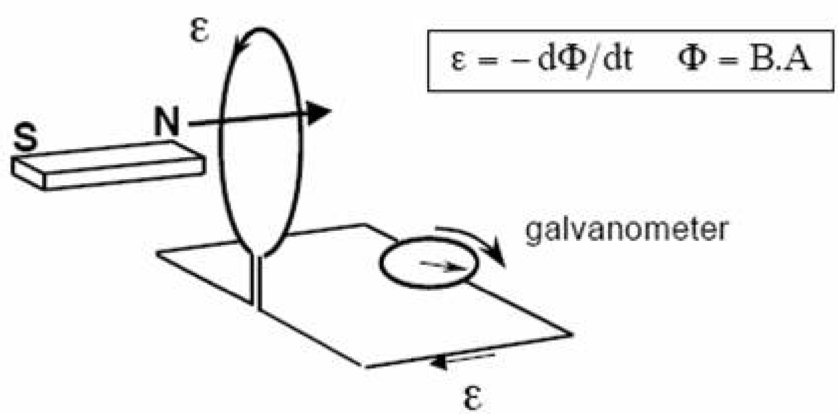


Figure 10.2:

The figure below shows the case of e.m.f induced when a permanent magnet is moved into a loop of wire

10.0.2 Motional e.m.f:

please refer to the figure given below. The magnetic flux density B is costant (in space and in time) and is normal to the plane containing the closed path.

Let the shorting bar be moving with a velocity v m/s. Let the bar move a small distance dl in time dt. then $dl = vdt$. Then the differential flux change is given by $d\phi = BvLdt$. The magnitude of the e.m.f induced is equal to

$$v_{ind} = -\frac{d\phi}{dt} = -BLv \tag{10.8}$$

In the general case where the direction of the movement of the conductor and the direction of the flux is such that they are not

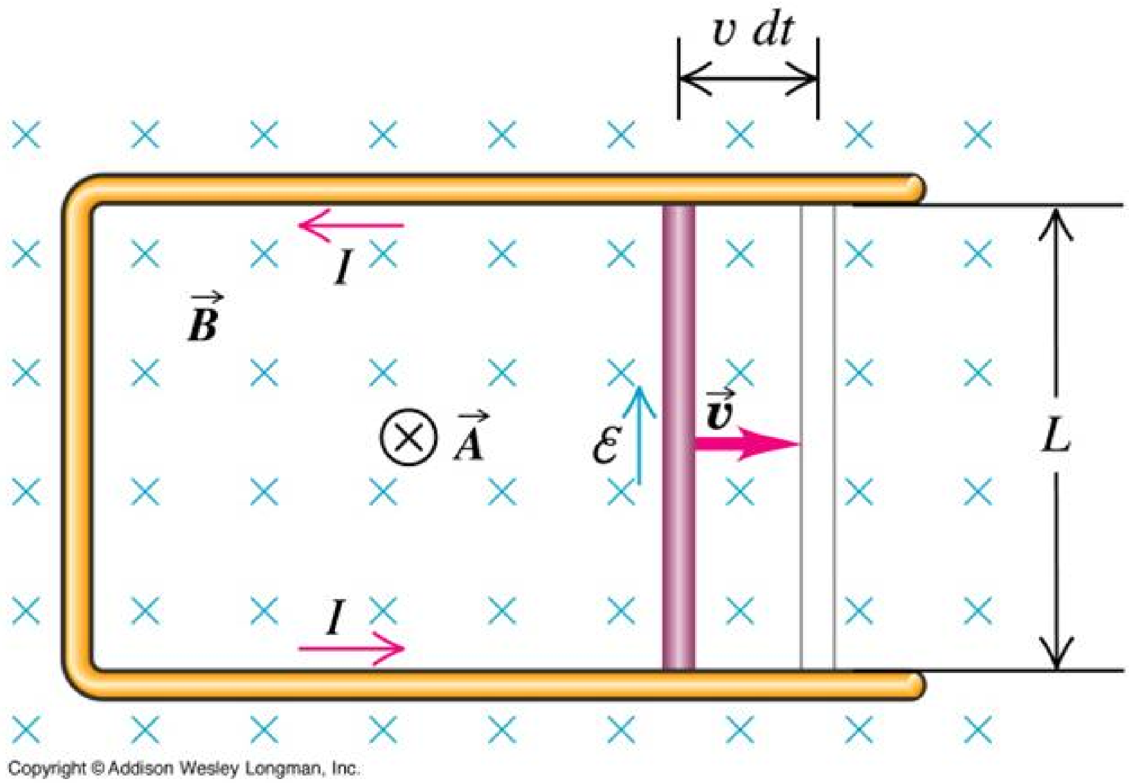


Figure 10.3:

orthogonal to each other then the emf induced is given in both magnitude and direction by

$$v_{ind} = \oint (v \times B) \bullet dl \tag{10.9}$$

Lenz's Law: The polarity of th induced e.m.f is given by Lenz's law. The Lenz's law states that the induced voltage acts to produce a flux which will try to oppose the original flux change which is the cause of production of the e.m.f. Th e following figures show the application of Lenz's law under various situations.

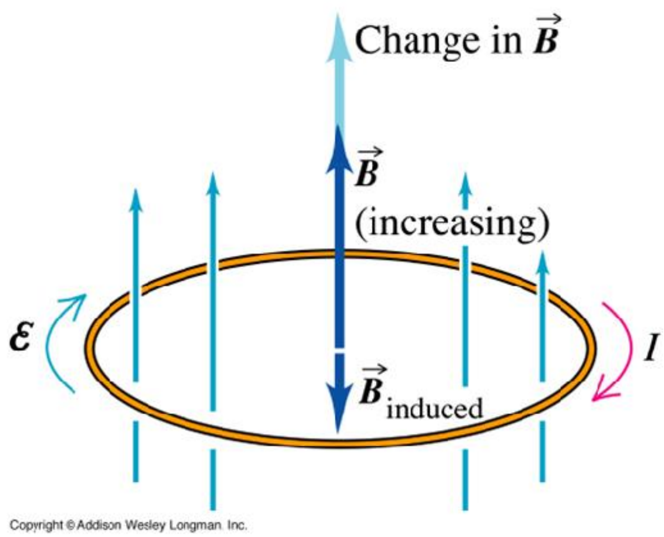


Figure 10.4:

The figure below shows different situations and the application of the Lenz's law

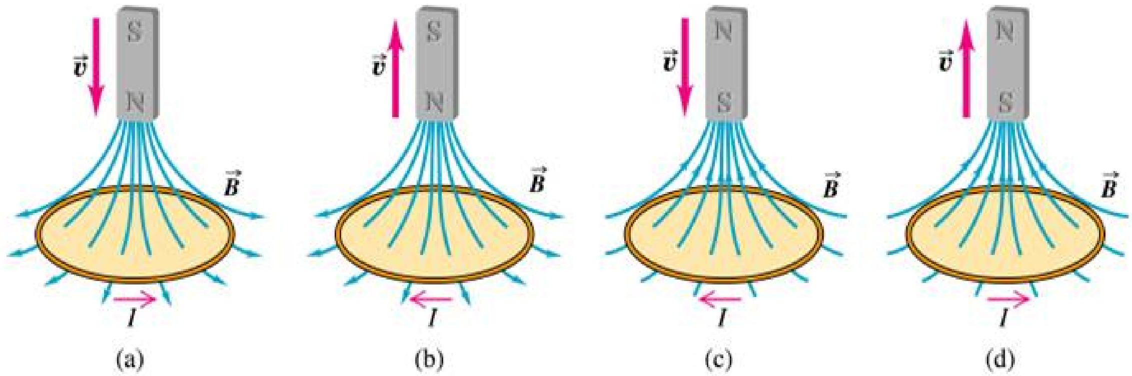


Figure 10.5:

The figure below is another way of looking at Lenz's law

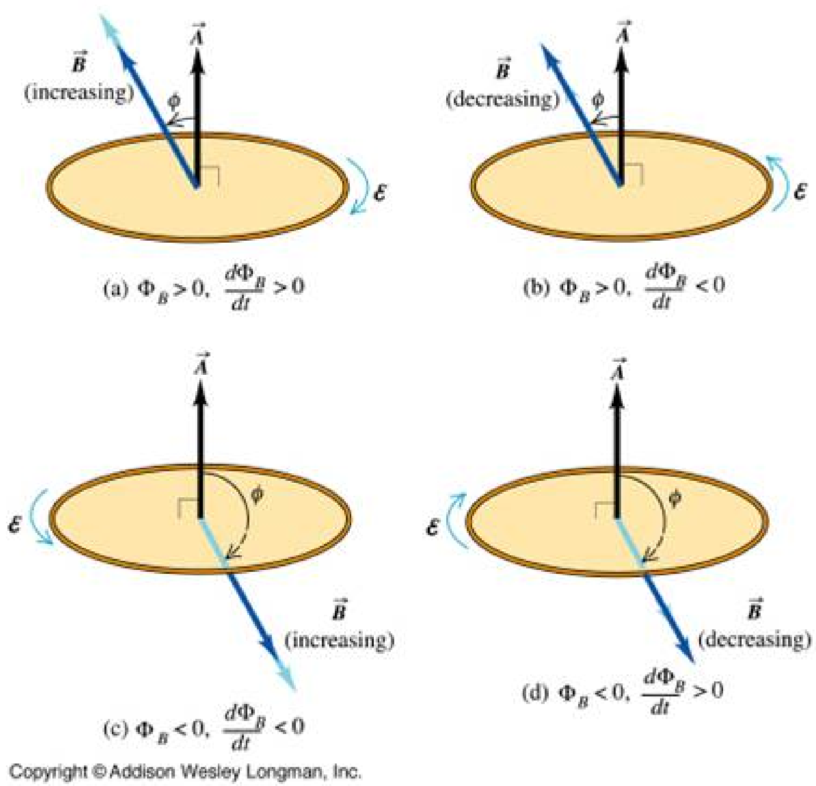


Figure 10.6: Different Situations And The E.M.F Induced

10.0.3 Displacement Current density:

Faraday's law as one of Maxwell's equation in differential form is given by

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

which shows that time cahanging magnetic field produces an electric field. This electric field has the special property that its line integral around a closed path is not zero. let us see what happens when a time changing electric field is considered. Consider the point form of the Ampere's circuital law

$\nabla\times H = J$ and see what happens when we take the divergence$\nabla\bullet\nabla\times H = 0 = \nabla\bullet J$ (10.10)

Since the divergence of curl is identically zero $\nabla \bullet J = 0$ However the equation of continuity shows that it can be true only if

$$\frac{\partial \rho_v}{\partial t} = 0 \tag{10.11}$$

This is an unrealistic limitation and the formula $\nabla \times H = J$ must be ammended .

Suppose we add an unknown term G to $\nabla \times H = J$, then the equation becomes

$$\nabla \times H = J + G \tag{10.12}$$

Again taking the divergence we have

$$0 = \nabla \bullet J + \nabla \bullet G \tag{10.13}$$

Thus

$$\nabla \bullet G = \frac{\partial \rho_v}{\partial t} \tag{10.14}$$

Replacing ρ_v by $\nabla \bullet D$

$$\nabla \bullet G = \frac{\partial}{\partial t}(\nabla \bullet D) = \nabla \bullet \frac{\partial D}{\partial t} \tag{10.15}$$

from which we obtain for G as

$$G = \frac{\partial D}{\partial t} \tag{10.16}$$

Ampere's circuital law in point from then becomes

$$\nabla \times H = J + \frac{\partial D}{\partial t} \tag{10.17}$$

The above equation is not derived. It is merely a form that was obtained which does not disagree with the continuity equation. It is also consistent all other results. The additional term $\frac{\partial D}{\partial t}$ has the dimensions of current density , Amp/square meter. Since it results from a time varying electrical flux density (or displacement density), this is called as displacement current density . It is sometimes denoted by J_d

$$\nabla \times H = J + J_d$$
$$J_d = \frac{\partial D}{\partial t}$$

we have encountered three types of current density they are

Conduction current density

$$J = \sigma E \tag{10.18}$$

Convection current density

$$J = \rho_v v \tag{10.19}$$

Displacement current density

$$J_d = \frac{\partial D}{\partial t} \tag{10.20}$$

The total displacement current crossing any given surface is expressed by the surface integral

$$I_d = \int_s J_d \bullet ds = \int_s \frac{\partial D}{\partial t} \bullet ds \tag{10.21}$$

and this leads to the time-vaying version of the Ampere's circuital law

$$\int_s (\nabla \times H) \bullet ds = \int_s J \bullet ds + \int_s J_d \bullet ds = \int_s J \bullet ds + \int_s \frac{\partial D}{\partial t} \bullet ds \tag{10.22}$$

and applying Stoke's theorem

$$\oint H \bullet dl = I + I_d = I + \int_s \frac{\partial D}{\partial t} \bullet ds \tag{10.23}$$

What is the nature of displacement current density? Let us study the simple circuit shown in the figure.

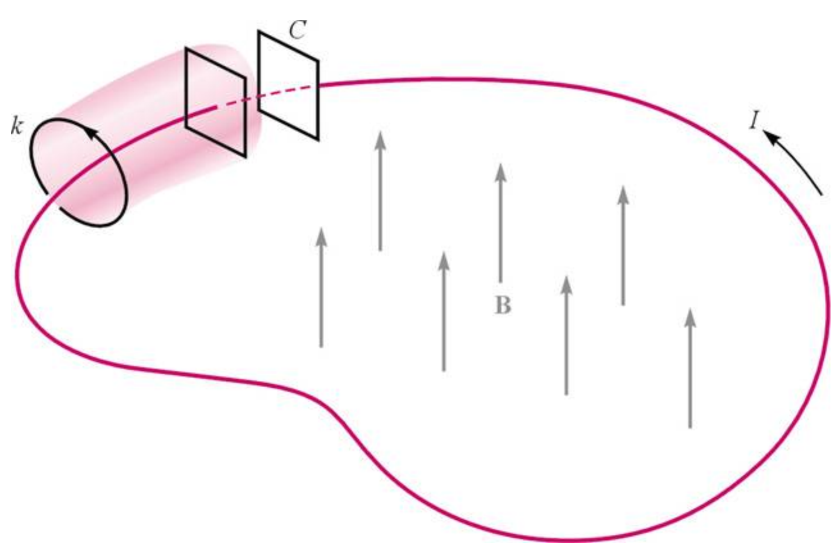


Figure 10.7: Filamentary Conducting Loop In A Time-varying Magnetic Field

It contains a filamentary loop and a parallel plate capacitor. With the loop a magnetic field varying sinusoidally with time is applied to produce an e.m.f about the closed path (the filament plus the dashed portion between the capacitor plates) which we shall take as

$$e.m.f = V_0 \cos \omega t \tag{10.24}$$

Using elementary circuit theory concepts and assuming that the loop has negligible resistance and inductance, we may obtain the current in the loop as

$$\begin{aligned} I &= -\omega C V_0 \cos \omega t \\ I &= -\omega \frac{\epsilon S}{d} \sin \omega t \end{aligned}$$

where the quantities ϵ, S, d pertain to the capacitor. Let us apply Ampere's circuital law about the smaller closed circular path k and neglect the displacement current for the moment

$$\oint_k H \bullet dl = I_k \tag{10.25}$$

The path and the value of H along the path are both definite quantities (although difficult to determine), and $\oint_k H \bullet dl$ is a definite quantity. The current I_k is that current through every surface whose perimeter is the path k . If we choose a simple surface punctured by the filament , such as the plane circular surface defined by the circular path k , the current is evidently the conduction current . Suppose now we consider the closed path k as the mouth of a paper bag whose bottom passes between the capacitor plates. The bag is not pierced by the filament, and the conduction current is zero. Now we need to consider the displacement current, for within the capacitor

$$D = \epsilon E = \epsilon \left(\frac{V_0}{d} \cos \omega t \right) \tag{10.26}$$

and therefore

$$I_d = -\omega \frac{\epsilon s}{d} V_0 \sin \omega t \tag{10.27}$$

This is the same value as that of the conduction current in the filamentary loop. The application of the Ampere's circuital law including the displacement current to the path k leads to a definite value for the line integral of H . This value must be equal to the total current crossing the chosen surface . For some surfaces the current is almost entirely conduction current, but for those surfaces passing between the capacitor plates, the conduction current is zero , and it is the displacement current which is now equal to the closed line integral of H .

Displacement current is associated with time varying electric fields and therefore exists in all imperfect conductors carrying time-varying conduction current. The reason why this additional current was never discovered experimentally is, it is very very small compared to the conduction current.

Maxwell's Equations

Static Fields	
Point or Differential Form	*Integral Form*
$$\begin{aligned} \nabla \bullet \mathbf{D} &= \rho_v \\ \nabla \times \mathbf{E} &= 0 \\ \nabla \bullet \mathbf{B} &= 0 \\ \nabla \times \mathbf{H} &= \mathbf{J}_c \end{aligned}$$	$$\begin{aligned} \oint_s \mathbf{D} \bullet d\mathbf{s} &= \int_v \rho_v dv \\ \oint_l \mathbf{E} \bullet d\mathbf{l} &= 0 \\ \oint_s \mathbf{B} \bullet d\mathbf{s} &= 0 \\ \oint_l \mathbf{H} \bullet d\mathbf{l} &= \oint_s \mathbf{J}_c \bullet d\mathbf{s} \end{aligned}$$

Maxwell's Equations

Time varying Fields	
Point or Differential Form	*Integral Form*
$\nabla \bullet \mathbf{D} = \rho_v$ $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ $\nabla \bullet \mathbf{B} = 0$ $\nabla \times \mathbf{H} = \mathbf{J}_c + \frac{\partial \mathbf{D}}{\partial t}$	$\oint_s \mathbf{D} \bullet ds = \int \rho_v dv$ $\oint_l \mathbf{E} \bullet dl = -\int_s \frac{\partial \mathbf{B}}{\partial t} \bullet ds$ $\oint_s \mathbf{B} \bullet ds = 0$ $\oint_l \mathbf{H} \bullet dl = \oint_s \mathbf{J} \bullet ds + \oint_s \frac{\partial \mathbf{D}}{\partial t} \bullet ds$

Maxwell's equations represent mathematical expressions of certain experimental results. In this light it is apparent that they cannot be proved; however , the applicability to any situation can be verified. As a result of extensive experimental work, Maxwell's equations are now believed to apply to all macroscopic situationsand they are used, much like conservation of momentum, as guiding principles. They are the fundamental equations of the electromagnetic fields produced by the source charges and current densities ρ and $\mathbf{J}$. If the material bodies are present , in order to use the Maxwell's equations, one must also know the applicable constitutive equations - either experimentally or from a microscopic theory of the particular kind of matter; $\mathbf{D} = \mathbf{D}(\mathbf{E})$ and $\mathbf{H} = \mathbf{H}(\mathbf{B})$

Maxwell's integral laws encompass the laws of electrical circuits. The transition from fields to circuits is made by associating the relevant volumes, surfaces, and contours with electrodes, wires, and terminal pairs. Begun in an informal way in Chap. 1, this use of the integral laws will be formalized and examined as the following chapters unfold. Indeed, many of the empirical origins of the integral laws are in experiments involving electrodes, wires and the like.

The remarkable fact is that the integral laws apply to any combination of volume and enclosing surface or surface and enclosing contour, whether associated with a circuit or not. This was implicit in our use of the integral laws for deducing field distributions in Chap. l.

Even though the integral laws can be used to determine the fields in highly symmetric configurations, they are not generally applicable to the analysis of realistic problems. Reasons for this lie beyond the geometric complexity of practical systems. Source distributions are not generally known, even when materials are idealized as insulators and "perfect" conductors. In actual materials, for example, those having finite conductivity, the self-consistent interplay of fields and sources, must be described.

Because they apply to arbitrary volumes, surfaces, and contours, the integral laws also contain the differential laws that apply at each point in space. The differential laws derived in this chapter provide a more broadly applicable basis for predicting fields. As might be expected, the point relations must involve information about the shape of the fields in the neighborhood of the point. Thus it is that the integral laws are converted to point relations by introducing partial derivatives of the fields with respect to the spatial coordinates.

As a description of the temporal evolution of electromagnetic fields in three-dimensional space, Maxwell's equations form a concise summary of a wider range of phenomena than can be found in any other discipline. Maxwell's equations are an intellectual achievement that should be familiar to every student of physical phenomena. As part of the theory of fields that includes continuum mechanics, quantum mechanics, heat and mass transfer, and many other disciplines, our subject develops the mathematical language and methods that are the basis for these other areas.

To quote Richard Feynman

> " From a long view of mankind - seen from, say, ten thousand years from now - there can be little doubt that the most significant event of the $19\,th$ century will be judged as Maxwell's discovery of the laws of electrodynamics. The American civil war will pale into insignificance in comparison with this important scientific event of the same decade"

It took the genius of James Clerk Maxwell to unify electricity and magnetism into a super theory, electromagnetism or classical electrodynamics (CED), and to realize that optics is a sub-field of this new super theory. Early in the 20th century, Nobel laureate Hendrik Antoon Lorentz took the electrodynamics theory further to the microscopic scale and also laid the foundation for the special theory of relativity, formulated by Albert Einstein in 1905. In the 1930s Paul A. M. Dirac expanded electrodynamics to a more symmetric form, including magnetic as well as electric charges and also laid the foundation for the development of quantum electrodynamics (QED).

Maxwell has made one of the great unifications of physics. Before his time, there was light, and there was electricity and magnetism. The latter two had been unified by the experimental work of Faraday, Oerstead, and ampere. Then, all of a sudden, light was no longer " something else," but was only electricity and magnetism in the new form - little pieces of electric and magnetic fields which propagate through space on their own.

The first equation - that the divergence of **E** is the charge density over ϵ_0 - is true in general. In dynamic as well as in static fields , Gauss' law is always valid. The flux of **E** through any closed surface is proportional to the charge inside. The third equation is the corresponding general law for magnetic fields. Since there are no magnetic charges, the flux of **B** through any closed surface is always zero. The second equation that the curl of **E** is $-\frac{\partial \mathbf{B}}{\partial t}$, is Faraday's law and was discussed . It is also generally true . The last equation has something new . We have seen before the part of it which holds for steady currents. In that case we said that the curl of **B** is $\mu_0 \mathbf{J}$, but the correct general expression has a new part that was discovered by Maxwell.

Maxwell's equations are as important today as ever. They led to the development of special relativity and, nowadays, almost every optics problem that can be formulated in terms of dielectric permittivity and magnetic permeability (two key constants in Maxwell's equations), ranging from optical fiber waveguides to meta-materials and transformation optics , is treated within the framework of these equations or systems of equations derived from them. Their actual solution can, however, be challenging for all but the most basic physical geometries. Numerical methods for solving the equations were pioneered by Kane Yee and Allen Taflove, but went unnoticed for many years owing to the limited computing power available at the time. Now, however, these methods can be easily employed for solving electromagnetic problems for structures as complex as aircraft. By the middle of the nineteenth century, a significant body of experimental and theoretical knowledge about electricity and magnetism had been accumulated. In 1861, James Clerk Maxwell condensed it into 20 equations. Maxwell published various reduced and simplified forms, but Oliver Heaviside is frequently credited with simplifying them into the modern set of four partial differential equations: Faraday's law, Ampère's law, Gauss' law for magnetism and Gauss' law for electricity. One of the most important contributions made by Maxwell was actually a correction to Ampère's law. He had realized that magnetic fields can be induced by changing electric fields — an insight that was not only necessary for accuracy but also led to a conceptual breakthrough. Maxwell predicted an 'electromagnetic wave', which can self-sustain, even in a vacuum, in the absence of conventional currents. Moreover, he predicted the speed of this wave to be 310,740,000 m/s — within a few percent of the exact value of the speed of light. "The agreement of the results seems to show that light and magnetism are affections of the same substance, and light is an electromagnetic disturbance propagated through the field according to electromagnetic laws", wrote Maxwell in 1865. The concept of light was thus unified with electricity and magnetism for the first time.

Maxwell's Equations

Static Fields	
Point or Differential Form	*Integral Form*
$$\begin{aligned} \nabla \bullet \mathbf{D} &= \rho_v \\ \nabla \times \mathbf{E} &= 0 \\ \nabla \bullet \mathbf{B} &= 0 \\ \nabla \times \mathbf{H} &= \mathbf{J}_c \end{aligned}$$	$$\begin{aligned} \oint_s \mathbf{D} \bullet d\mathbf{s} &= \int_v \rho_v dv \\ \oint_l \mathbf{E} \bullet d\mathbf{l} &= 0 \\ \oint_s \mathbf{B} \bullet d\mathbf{s} &= 0 \\ \oint_l \mathbf{H} \bullet d\mathbf{l} &= \oint_s \mathbf{J}_c \bullet d\mathbf{s} \end{aligned}$$

Maxwell's Equations

Time varying Fields	
Point or Differential Form	*Integral Form*
$\nabla \bullet \mathbf{D} = \rho_v$ $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ $\nabla \bullet \mathbf{B} = 0$ $\nabla \times \mathbf{H} = \mathbf{J}_c + \frac{\partial \mathbf{D}}{\partial t}$	$\oint_s \mathbf{D} \bullet ds = \int \rho_v dv$ $\oint_l \mathbf{E} \bullet dl = -\int_s \frac{\partial \mathbf{B}}{\partial t} \bullet ds$ $\oint_s \mathbf{B} \bullet ds = 0$ $\oint_l \mathbf{H} \bullet dl = \oint_s \mathbf{J} \bullet ds + \oint_s \frac{\partial \mathbf{D}}{\partial t} \bullet ds$

Index

Bibliography

[1] L.J.Chu Adler, R.B. and R.M.Fano.

[2] Kuc.R.

[3] Shafer.R.W Oppenhiem.A.v.